LECTURE NOTES ON DYNAMICAL SYSTEMS, CHAOS AND FRACTAL GEOMETRY

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كلية التربية للعلوم الصرفة

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Preface

These notes were created over a number of years of teaching senior seminar type courses to senior year students at Towson University and graduate courses in the Towson University Applied Mathematics Graduate Program. Originally I based the course on some of the existing text books currently available, such as [49], [14], [18] and [12], but gradually realized that these didn't fully suit the needs of our students, so I produced my own lecture notes. However, these notes owe a debt to the above mentioned books. Some of this material owes a debt to quite recent publications in the field of dynamical systems appearing in journals such as the *American Mathematical Monthly*, the *College Mathematics Journal*, *Mathematical Intelligencer* and *Mathematics Magazine*. Various internet resources have been used such Wolfram's MathWorld, some of these without citation because of difficulties in knowing who the author is. All of the figures were created using the computer algebra system Mathematica. Computer algebra systems are indispensable tools for studying all aspects of this subject. We often use Mathematica to simplify complicated algebraic manipulations, but avoid its use where possible. These notes are still a work in progress and will be regularly updated - many of the sections are still incomplete. I welcome any comments for improvement and would like to hear about any typographical or mathematical errors.

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Geoffrey R. Goodson

Dynamical Systems and Chaos: Spring 2013

CONTENTS

Chapter 1. The Orbits of One-Dimensional Maps.

1.1 Iteration of Functions and Examples of Dynamical Systems.

Dynamical systems is the study of how things change over time. Examples include the growth of populations, the change in the weather, radioactive decay, mixing of liquids and gases such as the ocean currents, motion of the planets, the interest in a bank account. Ideally we would like to study these with continuously varying time (what are called *flows*), but in this text we will simplify matters by only considering discrete changes in time. For example, we might model a population by measuring it daily. Suppose that x_n is the number of members of a population on day n, where x_0 is the initial population. We look for a function $f : \mathbb{R} \to \mathbb{R}$, (where $\mathbb{R}$ = set of all real numbers), for which

$$x_1 = f(x_0), x_2 = f(x_1) \quad \text{and generally} \quad x_n = f(x_{n-1}), \quad n = 1, 2, \ldots .$$

This leads to the iteration of functions in the following way:

Definition 1.1.1 Given $x_0 \in \mathbb{R}$, the *orbit* of x_0 under f is the set

$$O(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\},$$

where $f^2(x_0) = f(f(x_0))$, $f^3(x_0) = f(f^2(x_0))$, and continuing indefinitely, so that

$$f^n(x) = f \circ f \circ f \circ \cdots \circ f(x); \quad (n\text{-times composition}).$$

Set $x_n = f^n(x_0)$, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, so that in general

$$x_{n+1} = f^{n+1}(x_0) = f(f^n(x_0)) = f(x_n).$$

More generally, f may be defined on some subinterval I of $\mathbb{R}$, but in order for the iterates of $x \in I$ under f to be defined, we need the range of f to be contained in I, so $f : I \to I$.

This is what we call the iteration of one-dimensional maps (as opposed to higher dimensional maps $f : \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$, which will be studied briefly in a later chapter).

Definition 1.1.2 A (one dimensional) *dynamical system* is a function $f : I \to I$ where I is some subinterval of $\mathbb{R}$.

Given such a function f, equations of the form $x_{n+1} = f(x_n)$ are examples of *difference equations*. These arise in the types of examples we mentioned above. For

example in biology x_n may represent the number of bacteria in a culture after n hours. There is an obvious correspondence between one-dimensional maps and these difference equations. For example, a difference equation commonly used for calculating square roots: of 2 a in general $a > 0$

$$x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}),$$

corresponds to the function $f(x) = \frac{1}{2}(x + \frac{2}{x})$. If we start by setting $x_0 = 2$ (or in fact any real number), and then find x_1, x_2 etc., we get a sequence which rapidly approaches $\sqrt{2}$ (see page 9 of Sternberg [49]). One of the issues we examine is what exactly is happening here.

Examples of Dynamical Systems 1.1.3

1. **The Trigonometric Functions** Consider the iterations of the trigonometric functions starting with $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$. Select $x_0 \in \mathbb{R}$ at random, e.g., $x_0 = 2$ and set $x_{n+1} = \sin(x_n)$, $n = 0, 1, 2, \ldots$. Can you guess what happens to x_n as n increases? One way to investigate this type of dynamical system is to enter 2 into our calculator, then repeatedly press the **2nd**, **Answer** and **Sin** keys (you will need to do this many times to get a good idea - it may be easier to use Mathematica, or some similar computer algebra system). Do the same, but replace the sine function with the cosine function. How do we explain what appears to be happening in each case? These are questions that we aim to answer quite soon.

2. **Linear Maps** Probably the simplest dynamical system (and least interesting from a chaotic dynamical point of view), for population growth arises from the iteration of *linear maps*: maps of the form $f(x) = a \cdot x$. Suppose that x_n = size of a population at time n, with the property

$$x_{n+1} = a \cdot x_n$$

for some constant $a > 0$. This is an example of a *linear model* for the growth of the population.

If the initial population is $x_0 > 0$, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and in general $x_n = a^n x_0$ for $n = 0, 1, 2, \ldots$. This is the exact solution (or closed form solution) to the difference equation $x_{n+1} = a \cdot x_n$. Clearly $f(x) = ax$ is the corresponding dynamical system. We can use the solution to determine the long term behavior of the population:

x_n is very well behaved since:

(i) if $a > 1$, then $x_n \to \infty$ as $n \to \infty$,

(ii) if $0 < a < 1$ then $x_n \to 0$ as $n \to \infty$ (i.e., the population becomes extinct),

(iii) if $a = 1$, then the population remains unchanged.

3. **Affine maps** These are functions $f : \mathbb{R} \to \mathbb{R}$ of the form $\boxed{f(x) = ax + b}$ $(a \neq 0)$, for constants a and b. Consider the iterates of such maps:

$$f^2(x) = f(ax+b) = a(ax+b) + b = a^2x + ab + b,$$

$$f^3(x) = a^3x + a^2b + ab + b,$$

$$f^4(x) = a^4x + a^3b + a^2b + ab + b,$$

and generally

$$f^n(x) = a^nx + a^{n-1}b + a^{n-2}b + \cdots + ab + b.$$

Let $x_0 \in \mathbb{R}$ and set $x_n = f^n(x_0)$, then we have

$$\begin{aligned} x_n &= a^nx_0 + (a^{n-1} + a^{n-2} + \cdots + a + 1)b \\ &= a^nx_0 + b\left(\frac{a^n - 1}{a - 1}\right), \quad \text{if } a \neq 1, \end{aligned}$$

or

$$\boxed{x_n = \left(x_0 + \frac{b}{a-1}\right)a^n + \frac{b}{1-a},} \quad \text{if } a \neq 1,$$

is the closed form solution in this case (here we have used the formula for the sum of a finite geometric series:

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1},$$

when $r \neq 1$). If $a = 1$, the solution is $x_n = x_0 + nb$.

We can use these equations to determine the long term behavior of x_n. We see that:

(i) if $|a| < 1$ then $a^n \to 0$ as $n \to \infty$, so that

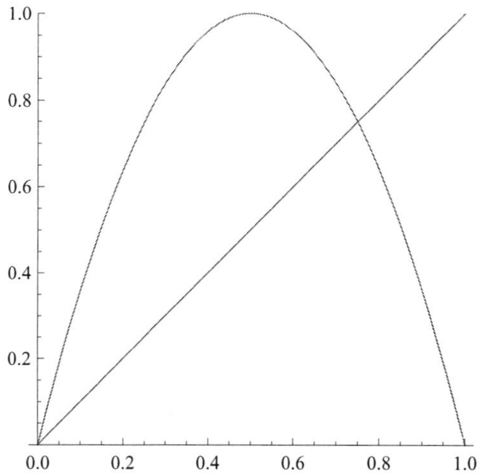
$$\lim_{n\to\infty} x_n = \frac{b}{1-a},$$

(ii) if $a > 1$, then $\lim_{n\to\infty} x_n = \infty$ for $b, x_0 > 0$,

(iii) if $a = 1$, then $\lim_{n\to\infty} x_n = \infty$ if $b > 0$.

The limit won't exist if $a \leq -1$.

4. **The Logistic Map** $L_\mu : \mathbb{R} \to \mathbb{R}$, $L_\mu(x) = \mu x(1-x)$ was introduced to model a certain type of population growth (see [29]). Here μ is a real parameter which is fixed. Note that if $0 < \mu \leq 4$ then L_μ is a dynamical system of the interval $[0, 1]$, i.e. $L_\mu : [0, 1] \to [0, 1]$). For example, when $\mu = 4$, $L_4(x) = 4x(1-x)$, with $L_4([0, 1]) = [0, 1]$ with graph given below. If $\mu > 4$, then L_μ is no longer a dynamical system of $[0, 1]$ as $L_\mu([0, 1])$ is not a subset of $[0, 1]$.



Recurrence Relations 1.1.4

Many sequences can be defined *recursively* by specifying the first term (or two), and then stating a general rule which specifies how to obtain the nth term from the $(n-1)$th term (or other additional terms), and using mathematical induction to see that the sequence is "well defined" for every $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For example, $n! = n$-factorial can be defined in this way by specifying $0! = 1$, and $n! = n \cdot (n-1)!$, for $n \in \mathbb{Z}^+$. The Fibonacci sequence F_n, can be defined by setting

$$F_0 = 1, \quad F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}, \quad \text{for} \quad n \in \mathbb{Z}^+,$$

so that $F_2 = 2$, $F_3 = 5$ etc.

The orbit of a point $x_0 \in \mathbb{R}$ under a function f is then defined recursively as follows: we are given the starting value x_0, and we set

$$x_n = f(x_{n-1}), \quad \text{for} \quad n \in \mathbb{Z}^+.$$

The principle of mathematical induction then tells that x_n is defined for every $n =\geq 0$, since it is defined for $n = 0$, and assuming it has been defined for $k = n-1$ then $x_n = f(x_{n-1})$ defines it for $k = n$.

Ideally, given a recursively defined sequence x_n, we would like to have a specific formula for x_n in terms of elementary functions (so called *closed form solution*), but this is often very difficult, or impossible to achieve. In the case of affine maps and certain logistic maps, there is a closed form solution. One can use the former to study problems of the following type:

Example 1.1.5 An amount \$ T is deposited in your bank account at the end of each month. The interest is $r\%$ per period. Find the amount $A(n)$ accumulated at the end of n months (Assume $A(0) = T$).

Answer. $A(n)$ satisfies the difference equation

$$A(n+1) = A(n) + A(n)r + T, \quad \text{where } A(0) = T,$$

or

$$A(n+1) = A(n)(1+r) + T,$$

so we set $x_0 = T$, $a = 1 + r$ and $b = T$ in the formula of Example 1.1.3, then the solution is

$$\begin{aligned} A(n) &= (1+r)^n T + T\left(\frac{(1+r)^n - 1}{1+r-1}\right) \\ &= (1+r)^n T + \frac{T}{r}((1+r)^n - 1). \end{aligned}$$

Remark 1.1.6 It is conjectured that closed form solutions for the difference equation arising from the logistic map are only possible when $\mu = \pm 2, 4$ (see Exercises 1.1 # 3 for the cases where $\mu = 2, 4$, and # 7 for the case where $\mu = -2$ and also [53] for a discussion of this conjecture).

Exercises 1.1

1. If $L_\mu(x) = \mu x(1-x)$ is the logistic map, calculate $L^2_\mu(x)$ and $L^3_\mu(x)$.

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2. Use Example 1.1.3 for affine maps to find the solutions to the difference equations:

(i) $x_{n+1} - \dfrac{x_n}{3} = 2$, $x_0 = 2$,

(ii) $x_{n+1} + 3x_n = 4$, $x_0 = -1$.

3. A logistic difference equation is one of the form $x_{n+1} = \mu x_n(1 - x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact solutions to the logistic equations:

(i) $x_{n+1} = 2x_n(1 - x_n)$. Hint: use the substitution $x_n = (1 - y_n)/2$ to transform the equation into a simpler equation that is easily solved.

(ii) $x_{n+1} = 4x_n(1 - x_n)$. Hint: set $x_n = \sin^2(\theta_n)$ and simplify to get an equation that is easily solved.

4. You borrow \$ P at r % per annum, and pay off \$ M at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100,000$, $M = 1000$ and $r = 4$, after how long will the loan be paid off?

5. If $T_\mu(x) = \begin{cases} \mu x & \text{if } 0 \leq x < 1/2 \\ \mu(1-x) & \text{if } 1/2 \leq x < 1 \end{cases}$, show that T_μ is a dynamical system of $[0,1]$ for $\mu \in (0,2]$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined below. For each of the intervals given, determine whether f can be considered as a dynamical system $f : I \to I$:

(a) $f(x) = x^3 - 3x$,

$$\text{(i)} \quad I = [-1,1], \quad \text{(ii)} \quad I = [-2,2].$$

(b) $f(x) = 2x^3 - 6x$,

$$\text{(i)} \quad I = [-1,1], \quad \text{(ii)} \quad I = [-\sqrt{\frac{7}{2}}, \sqrt{\frac{7}{2}}], \quad \text{(iii)} \quad I = [-4,4].$$

7. For the following functions, find $f^2(x)$, $f^3(x)$ and a general formula for $f^n(x)$:

$$\text{(i)} \quad f(x) = x^2, \quad \text{(ii)} \quad f(x) = |x+1|, \quad \text{(iii)} \quad f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1. \end{cases}$$

8. Use mathematical induction to show that if $f(x) = \dfrac{2}{x+1}$, then

$$f^n(x) = \frac{2^n(x+2) + (-1)^n(2x-2)}{2^n(x+2) - (-1)^n(x-1)}.$$

9*. Show that a closed form solution to the logistic difference equation when $\mu = -2$ is given by

$$x_n = \frac{1}{2}\left[1 - f\left[r^n f^{-1}(1-2x_0)\right]\right], \quad \text{where} \quad r = -2 \quad \text{and} \quad f(\theta) = 2\cos\left(\frac{\pi - \sqrt{3}\theta}{3}\right).$$

(Hint: Set $x_n = \dfrac{1 - f(\theta_n)}{2}$ and use steps similar to 3(ii)).

1.2 Newton's Method and Fixed Points

Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$, Newton's method often allows us to find good approximations to zeros of $f(x)$, i.e., approximate solutions to the equation $f(x) = 0$. The idea is to start with a first approximation x_0 and look at the tangent line to $f(x)$ at the point $(x_0, f(x_0))$. Suppose this line intersects the x-axis at x_1, then we can check that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad \text{if} \quad f'(x_0) \neq 0.$$

For reasons that we will make clear, it is often the case that x_1 is a better approximation to the zero $x = c$ than x_0 was. In other words, Newton's method is an algorithm for finding approximations to zeros of a function $f(x)$. The algorithm gives rise to a difference equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

where x_0 is a first approximation to a zero of $f(x)$. The corresponding real function is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \text{(the } \textit{Newton} \text{ function)}.$$

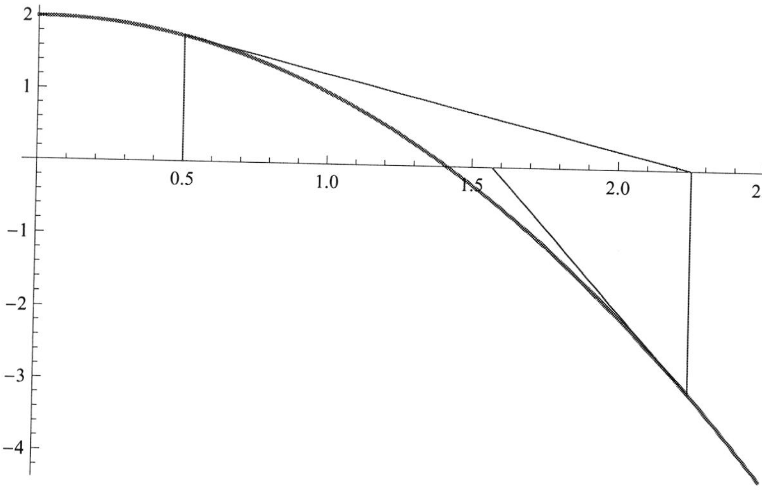
Note that if $f(x) = x^2 - a$, then $f'(x) = 2x$ and

$$N_f(x) = x - \frac{x^2 - a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right),$$

so that

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),$$

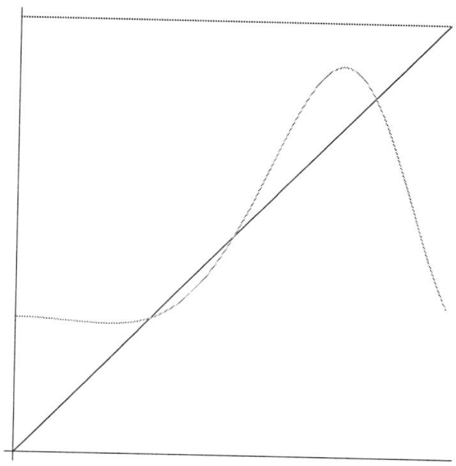
the difference equation we mentioned in Section 1.1 that is used for approximating $\sqrt{2}$ when $a = 2$.



The first two approximations for $f(x) = 2 - x^2$, starting with $x_0 = .5$.

Note that we are looking for where $f(x) = 0$, and this happens if and only if $N_f(x) = x$.

Definition 1.2.1 For a function $f : \mathbb{R} \to \mathbb{R}$, a point $c \in \mathbb{R}$ for which $f(c) = c$ is called a *fixed point* of f. It is a point where the graph of $f(x)$ intersects the line $y = x$. We denote the set of fixed points of f by $\text{Fix}(f)$.



Fixed points occur where the graph of $f(x)$ intersects the line $y = x$.

Example 1.2.2 Suppose that $f(x) = x^2$, then $x^2 = x$ gives $x(x-1) = 0$, so has fixed points $c = 0$ and $c = 1$, so $\text{Fix}(f) = \{0, 1\}$. If $f(x) = x^3 - x$, then $x^3 - x = x$ gives $x(x^2 - 2) = 0$, so the fixed points are $c = 0$ and $c = \pm\sqrt{2}$, $\text{Fix}(f) = \{0, \pm\sqrt{2}\}$.

The logistic map $f(x) = 4x(1-x) = 4x - 4x^2$, $0 \le x \le 1$ has the properties: $f(0) = 0$, (a fixed point), $f(1) = 0$, a maximum at $x = 1/2$ (with $f(1/2) = 1$). Solving $f(x) = x$ gives $4x - 4x^2 = x$, so $4x^2 = 3x$, so the fixed points are $c = 0$ and $c = 3/4$.

This map has what we call *eventual fixed points*: $f(1) = 0$ and $f(0) = 0$, so we say that $c = 1$ is eventually fixed. Also $f(1/4) = 3/4$, so $c = 1/4$ is eventually fixed, as is $c = (2+\sqrt{3})/4$.

Definition 1.2.3 $x^* \in \mathbb{R}$ is an *eventual fixed point* of $f(x)$ if there exists a fixed point c of $f(x)$ and $r \in \mathbb{Z}^+$ satisfying $f^r(x^*) = c$, but $f^s(x^*) \ne c$ for $0 < s < r$.

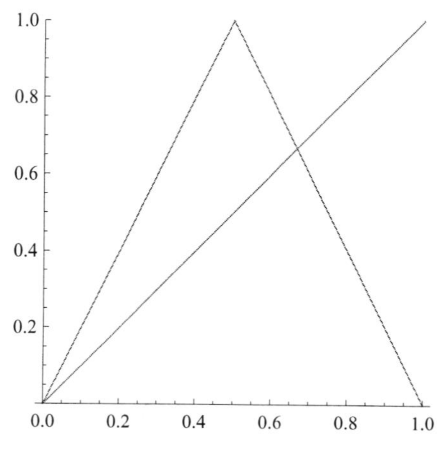
Example 1.2.4 The Tent Map

Define a function $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} 2x & : 0 \le x \le 1/2 \\ 2(1-x) & : 1/2 < x \le 1 \end{cases}$$

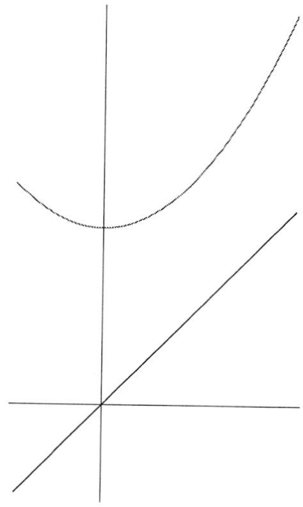
$$= 1 - 2|x - 1/2|.$$

$T(x)$ is called the *tent map*, it has fixed points: $T(0) = 0$ and $T(2/3) = 2/3$. Since $T(1/4) = 1/2$, $T(1/2) = 1$ and $T(1) = 0$, $1/4, 1/2$ and 1 are eventually fixed. It is not difficult to see that there are many other eventually fixed points (infinitely many).



Note that some maps do not have fixed points:

Example 1.2.5 $f(x) = x^2 + 1$ has no fixed points.

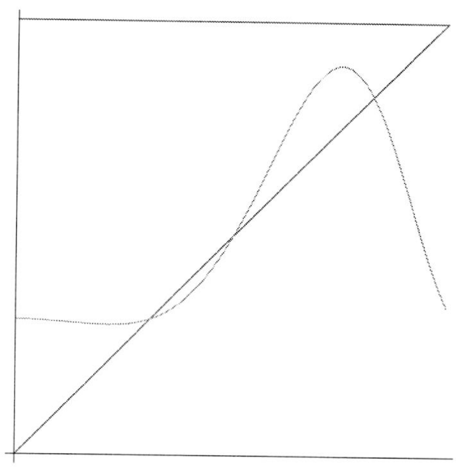


$f(x) = x^2 + 1$ does not intersect the line $y = x$.

Question 1.2.6 For what values of $a \in \mathbb{R}$ does $Q_a(x) = x^2 + a$ have fixed points? We see below that certain functions always have fixed points:

Theorem 1.2.7 *Let $f : I \to I$ be a continuous function, where $I = [a, b]$, $a < b$ is a bounded interval. Then $f(x)$ has a fixed point $c \in I$.*

Proof. Set $g(x) = f(x) - x$. We may assume that $f(a) \neq a$ and $f(b) \neq b$, so that $f(a) > a$ and $f(b) < b$.



The graph of $f(x)$ always intersects the line $y = x$.

It follows that

$$g(a) = f(a) - a > 0, \quad \text{and} \quad g(b) = f(b) - b < 0,$$

so that $g(x)$ is a continuous function which is positive at a and negative at b, so by the Intermediate Value Theorem (IVT), there exists $c \in (a, b)$ (open interval) with $g(c) = 0$, i.e., $f(c) = c$, so c is a fixed point of $f(x)$. $\square$

Remark 1.2.8 The above is an example of an existence theorem, it says nothing about how to find the fixed point, where it is or how many there are. It tells us that if $f(x)$ is a continuous function on an interval I with $f(I) \subseteq I$, then $f(x)$ has a fixed point in I. It is also true that if instead $f(I) \supseteq I$ then $f(x)$ has a fixed point in I as we see now:

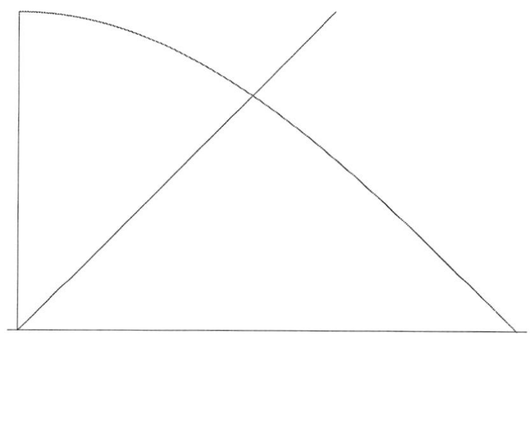
Theorem 1.2.9 *Let $f : I \to \mathbb{R}$ ($I = [a, b], a < b$) be a continuous function with $f(I) \supseteq I$, then $f(x)$ has a fixed point in I.*

Proof. As before, set $g(x) = f(x) - x$, then there exists $c_1 \in (a, b)$ with $f(c_1) < c_1$ (in fact $f(c_1) < a < c_1$). Also there is $c_2 \in (a, b)$ with $f(c_2) > c_2$.

Then $g(c_1) < 0$ and $g(c_2) > 0$ and since $g(x)$ is a continuous function, it follows by the Intermediate Value Theorem that there exists $c \in I$, ($c_1 < c < c_2$ or $c_2 < c < c_1$), with $g(c) = 0$; $f(c) = c$. $\square$

Remark 1.2.10 It is often difficult to find fixed points explicitly:

Example 1.2.11 Set $f(x) = \cos x$. If $f(c) = c$, then $\cos c = c$. It is possible to find an approximation to the fixed point $c = .739085\ldots$ using (for example) Newton's method.



$f(x) = \cos x$ has a fixed point in $[0, \pi/2]$.

Exercises 1.2

1. Find all fixed points and eventual fixed points of the map $f(x) = |x - 1|$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for some $n \in \mathbb{Z}^+$, the nth iterate of f has a unique fixed point c (i.e., $f^n(c) = c$ and c is unique). Then show that c is a fixed point of f.

3. Use the `Mathematica` command `NestList` (see below) to find how the following functions behave when the given point is iterated (comment on what appears to be happening in each case):

(i) $L_1(x) = x(1 - x)$, with starting point $x_0 = .75$.

(ii) $L_2(x) = 2x(1 - x)$, with starting point $x_0 = .1$.

(iii) $L_3(x) = 3x(1 - x)$, with starting point $x_0 = .2$.

(iv) $L_{3.2}(x) = 3.2x(1 - x)$, with starting point $x_0 = .95$.

(v) $f(x) = \sin(x)$, with starting point $x_0 = 9.5$.

(vi) $g(x) = \cos(x)$, with starting point $x_0 = -15.3$.

4. Show that the logistic map $L_\mu(x) = \mu x(1 - x)$, $x \in [0, 1]$, for $0 < \mu < 4$, the fixed point $x = 0$ has no eventual fixed points other than $x = 1$.

Show that for $1 < \mu \leq 2$, the fixed point $x = 1 - 1/\mu$ only has $x = 1/\mu$ as an eventual fixed point, but this is not the case for $2 < \mu < 3$.

To use `Mathematica`, *first define your function, say* $f(x) = \sin(x)$ *in an* `input cell` *using the syntax*

```
f[x_]:= Sin[x].
```

Execute the cell by doing `Shift` *and* `Enter` *simultaneously. The command*

```
NestList[f, 9.5, 100]
```

when executed in a new `input cell` *will give 100 iterations of* $\sin(x)$ *with starting value* $x_0 = 9.5$. *If you want to graph* $f(x)$ *on the interval* $[a, b]$, *use the command:*

```
Plot[f[x], { x, a, b } ].
```

5. (a) Let $f(x) = (1+x)^{-1}$. Find the fixed points of f and show that there are no points c with $f^2(c) = c$ and $f(c) \neq c$ (period 2-points). Note that $f(-1)$ is not defined, but points that get mapped to -1 belong to the interval $[-2, -1)$ and are of the form

$$\nu_n = -\frac{F_{n+1}}{F_n}, \quad n \geq 1,$$

where $\{F_n\}$, $n \geq 0$, is the Fibonacci sequence (see 1.1.4). Note that $\nu_n \to -r$ as $n \to \infty$, where $-r$ is the negative fixed point of f (see [7] for more details).

(b) If $x_0 \in (0, 1]$, set $x_n = \dfrac{F_{n-1}x_0 + F_n}{F_n x_0 + F_{n+1}}$. Use mathematical induction to show that

$$x_{n+1} = f(x_n) = \frac{F_n x_0 + F_{n+1}}{F_{n+1}x_0 + F_{n+2}}.$$

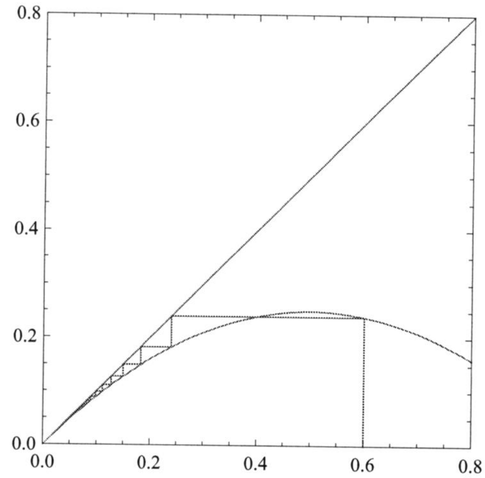
Deduce that as $n \to \infty$, $x_n \to 1/r$, the positive fixed point of f.

1.3 Graphical Iteration

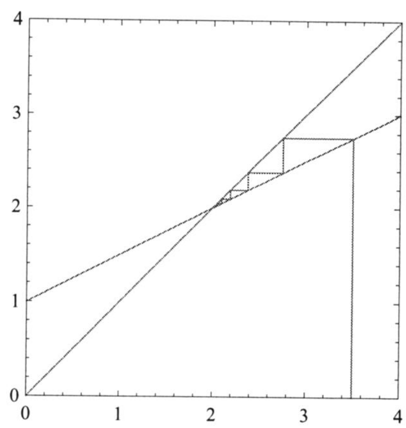
It is possible to follow the iterates of a function $f(x)$ at a point x_0 using *graphical iteration* (sometimes called *web diagrams*). or Cobweb diagrams or Stair Step

We start at x_0 on the x-axis and draw a line vertically to the function. We then move horizontally to the line $y = x$, then vertically to the function and continue in this way. Notice that in some examples the iterations converges to a fixed point. In others it goes off to ∞, whilst in others still, it oscillates between two points indefinitely.

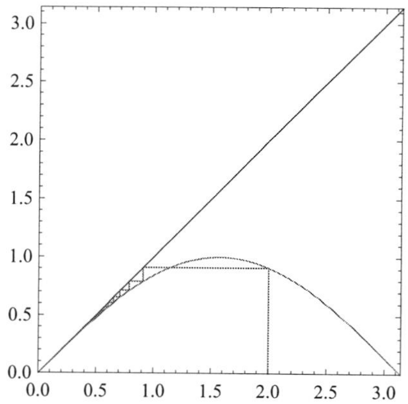
Example 1.3.1 $f(x) = x(1-x)$. In this example, an examination of graphical iteration seems to suggest that the orbits of any point in $[0, 1]$ approach the fixed point $x = 0$. This is an example of an *attracting* fixed point.



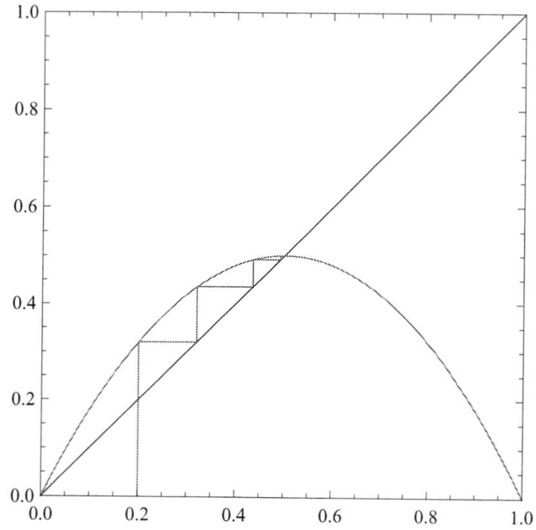
Example 1.3.2 Let $f(x) = x/2 + 1$. This is an affine transformation with $a = 1/2$, $b = 1$. According to what we saw in Example 1.1.3, the iterates should converge to $\frac{b}{1-a} = 2$ since $|a| < 1$. What is actually happening is that $c = 2$ is an attracting fixed point of $f(x)$ with the property that it attracts all members of $\mathbb{R}$ (said to be *globally attracting.*

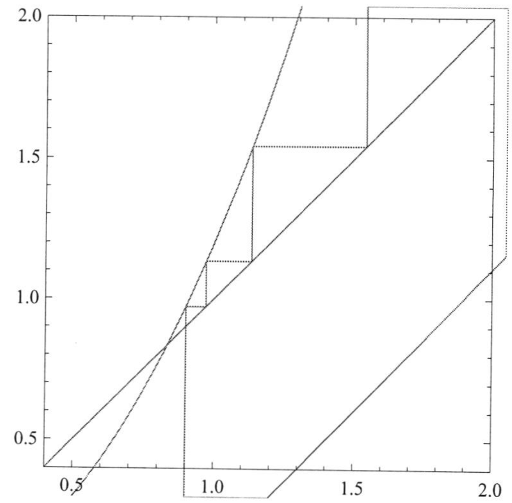


Example 1.3.3 We see from the graph of $f(x) = \sin x$ that $c = 0$ seems to be an attracting fixed point. We shall show later that it is actually globally attracting.



We see two basic situations arising (together with some variations of these). In particular, we notice that if the sequence $x_n = f^n(x_0)$ converges to some point c as $n \to \infty$, then c is a fixed point.
(i) Stable orbit, where the web diagram approaches a fixed point.
(ii) Unstable orbit, where the web diagram moves away from a fixed point.





Proposition 1.3.4 *If $y = f(x)$ is a continuous function and $\lim_{n\to\infty} f^n(x_0) = c$, then $f(c) = c$ (i.e., if the orbit converges to a point c, then c is a fixed point of f)* .

Proof. We see that

$$\lim_{n\to\infty} f^n(x_0) = c \Rightarrow f(\lim_{n\to\infty} f^n(x_0)) = f(c),$$

and since f is continuous,

$$\lim_{n\to\infty} f^{n+1}(x_0) = f(c).$$

But clearly $c = \lim_{n\to\infty} f^{n+1}(x_0)$, so that $c = f(c)$ by the uniqueness of limit. $\square$

1.4 Attractors and Repellers

Suppose that $f : X \to X$ is a function (where we assume that X is some subset of $\mathbb{R}$ such as an interval or possibly $\mathbb{R}$ itself). Let $c \in X$ be a fixed point of f. We define the notions on attracting fixed point and repelling fixed that we looked at graphically in the last section.

Definition 1.4.1 Let $f : X \to X$ with $f(c) = c$.

(f ممكن مستمرة أو ليست مستمرة)
إذا كانت f مستمرة فإن الجذب أقوى من الاستقرارية

(i) c is a *stable fixed point* if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ and $|x - c| < \delta$, then $|f^n(x) - c| < \epsilon$ for all $n \in \mathbb{Z}^+$.

If this does not hold, c will be called *unstable.*

(ii) c is said to be attracting if there is a real number $\eta > 0$ such that

$$|x - c| < \eta \Rightarrow \lim_{n\to\infty} f^n(x) = c.$$

(iii) c is *asymptotically stable* if it is both stable and attracting.

Remark 1.4.2 We will show that a fixed point c of $f : X \to X$ (suitably differentiable) with the property $|f'(c)| < 1$ is an asymptotically stable fixed point. If c is an unstable fixed point, we can find $\epsilon > 0$ and x arbitrarily close to c such that some iterate of x, say $f^n(x)$, will be greater than distance ϵ from c. This is the case when $|f'(c)| > 1$ as the next theorem shows. We call c a *repeller* (c is a *repelling fixed point*), since the iterates move away from the fixed point (c is unstable). We will also see that a fixed point can be stable without being attracting, and that it can be attracting without being stable.

Definition 1.4.3 A fixed point c of f is *hyperbolic* if $|f'(c)| \neq 1$. If $|f'(c)| = 1$ it is *non-hyperbolic.*

Thus if c is a non-hyperbolic fixed point, then $f'(c) = 1$ or $f'(c) = -1$, so the graph of $f(x)$ either meets the line $y = x$ tangentially, or at $90°$:

S. Ebaydi 2005

Thm ① A continuous map f on the real line cannot have an attracting unstable fixed point. البرهان في الأسفل بعد عرض Thm ②

Thm② A continuous map f is asymptotically stable if and only if fixed point c of a

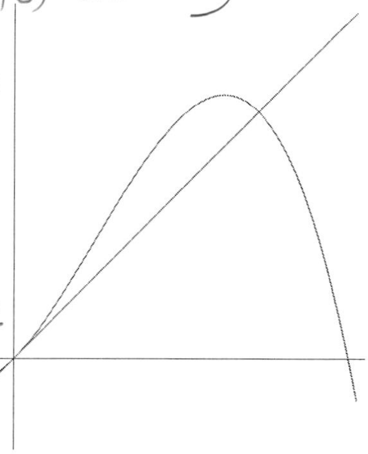
there is an open interval (a,b) containing c such that

$f^2(x) > x$ for $a < x < c$ and (البرهان غير مطلوب)

$f^2(x) < x$ for $c < x < b$.

Proof of ① let f cont. that has an unstable globally attracting f. pt. c

this implies that the equation $f^2(x) = x$ has only one sol. $x = c$



$f(x) = -2x^3 + 2x^2 + x$ has both types of non-hyperbolic fixed points.

اختبار للاستقرار المقارب (AS) في حالة f قد تكون غير قابلة للاشتقاق عند c

Criterion for asymptotic stability of f. pts. of nondiff. maps

Hyperbolic fixed points have the following properties:

Theorem 1.4.4 *Let $f : X \to X$ be a differentiable function with continuous first derivative (we say that f is of class C^1).*

(i) *If a is a fixed point for $f(x)$ with $|f'(a)| < 1$, then a is asymptotically stable. The iterates of points close to a, converge to a geometrically (i.e., there is a constant $0 < \lambda < 1$ for which $|f^n(x) - a| < \lambda^n |x - a|$ for all $n \in \mathbb{Z}^+$ and for all $x \in X$ sufficiently close to a).*

(ii) *If a is a fixed point for $f(x)$ for which $|f'(a)| > 1$, then a is a repelling fixed point for f.*

Proof. (i) We may assume that X is an open interval with $a \in X$. Suppose $|f'(a)| < \lambda < 1$ for some $\lambda > 0$, then using the continuity of $f'(x)$, there exists an open interval I with $|f'(x)| < \lambda < 1$ for all $x \in I$.

By the Mean Value Theorem, if $x \in I$ there exists $c \in I$, lying between x and a, satisfying

$$f'(c) = \frac{f(x) - f(a)}{x - a},$$

so that

$$|f(x) - a| = |f'(c)||x - a| < \lambda |x - a|,$$

(i.e., $f(x)$ is closer to a than x was).

Repeating this argument with $f(x)$ replacing x gives

$$|f^2(x) - a| < \lambda^2 |x - a|, \ldots, |f^n(x) - a| < \lambda^n |x - a|.$$

تتمة البرهان

(لأنه لو كان أحد يحقق $f^2(x_0) = x_0$ يعني ذلك أنه دوري ودورته 2-periodic فمساره لن يقترب إلى c)

Hence, there are only two possible cases

(AS) by② ⇐ (a) $f^2(x) > x$ for $x < x^*$ and $f^2(x) < x$ for $x > x^*$;

(b) $f^2(x) < x$ for $x < x^*$ and $f^2(x) > x$ for $x > x^*$. $\Rightarrow \cdots < f^4 < f^2 < x < x^*$ not conv.

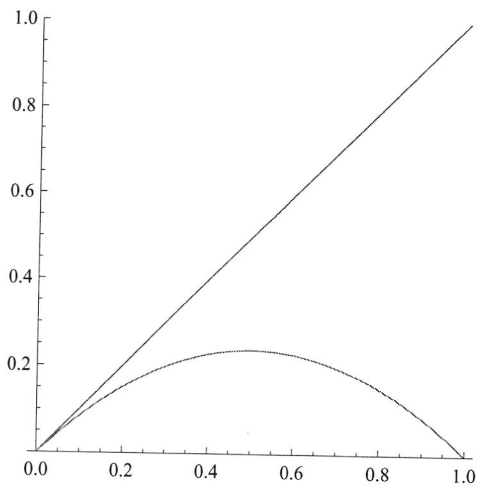
Since $\lambda^n \to 0$ as $n \to \infty$, it follows that $f^n(x) \to a$ as $n \to \infty$.

The proof of (ii) is similar. $\square$

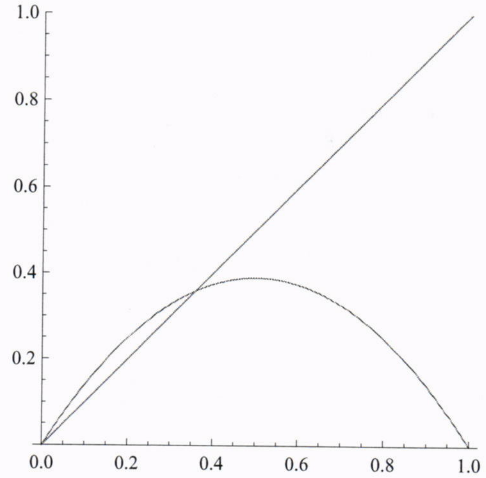
Example 1.4.5 Denote the Logistic map by $L_\mu(x) = \mu x(1-x) = \mu x - \mu x^2$. We are interested in this map for $0 \le x \le 1$ and $0 < \mu \le 4$, since for these parameter values μ, L_μ is a function which maps the interval $[0,1]$ into the interval $[0,1]$, so it necessarily has a fixed point and iterates stay inside $[0,1]$ (if $\mu > 4$, $L_\mu(x) > 1$ for some values of x in $[0,1]$ and further iterates will go to $-\infty$).

Solving $L_\mu(x) = x$ gives $x = 0$ or $x = 1 - 1/\mu$. If $0 < \mu \le 1$, then $1 - 1/\mu \le 0$, so $c = 0$ is the only fixed point in $[0,1]$ in this case.

For $0 < \mu \le 1$, $L'_\mu(x) = \mu - 2\mu x$, so $L'_\mu(0) = \mu$ and 0 is an attracting fixed point for $0 < \mu < 1$.



If $\mu > 1$, then 0 and $1 - 1/\mu$ are both fixed points in $[0,1]$, but now 0 is repelling.



Hyper. f.p. $|f'(c)| \neq 1$

Also,

$$L'_\mu(1 - 1/\mu) = \mu - 2\mu(1 - 1/\mu) = 2 - \mu,$$

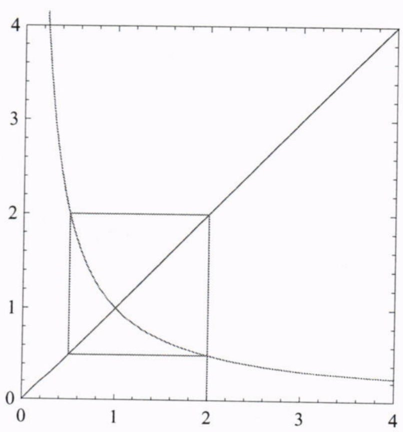
so that

$$|L'_\mu(1 - 1/\mu)| = |2 - \mu| < 1 \quad \text{iff} \quad 1 < \mu < 3.$$

In this case $c = 1 - 1/\mu$ is an attracting fixed point, and repelling for $\mu > 3$. Note that $L'_1(0) = 1$ and $L'_3(2/3) = 1$ so we have non-hyperbolic fixed points.

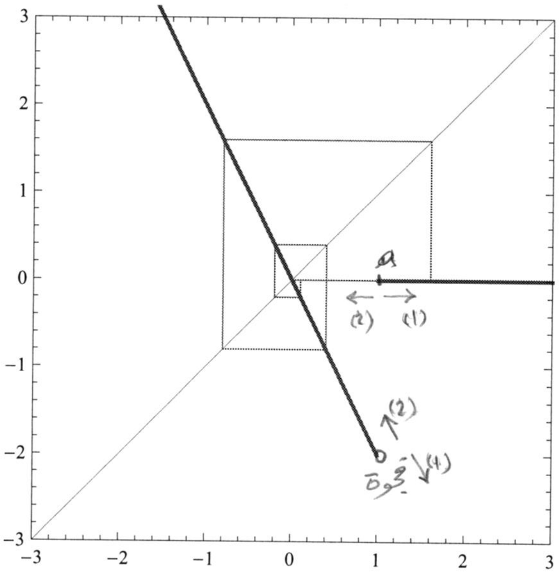
We will examine in Chapter 2 what happens in the cases where $\mu = 1$ and $\mu = 3$.

Example 1.4.6 Let $f(x) = 1/x$. The two fixed points $x = \pm 1$ are non- hyperbolic since $f'(x) = -1/x^2$, so $f'(\pm 1) = -1$. However, they are stable but not attracting since $f^2(x) = f(1/x) = x$, and we see points close to ± 1 neither move closer nor further apart.



 a>0

Example 1.4.7 Consider instead the function $f_a(x) = \begin{cases} -2x & \text{if } x < a \\ 0 & \text{if } x \geq a \end{cases}$, where a is any positive real number. Then $c = 0$ is an unstable (repelling) fixed point of f_a which is attracting i.e., $\lim_{n\to\infty} f^n(x) = 0$ for all $x \in \mathbb{R}$. A fixed point having this latter property is said to be *globally attracting.* Note that the function f_a is not continuous at $x = a$. It has been shown by Sedaghat [44] that a continuous mapping of the real line cannot have an unstable fixed point that is globally attracting.



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family

The map f_1 (in bold) where $x = 0$ is an attracting but unstable fixed point.

$f_a(x)$ not continuous

لذلك ممكن ان تكون النقطة الثابتة جذب لكن غير مستقرة . لكن في اي دالة مستمرة نقطة جذب إذا وجدت تكون مستقرة.

Example 1.4.8 Newton's Method Revisited.

Suppose that $f : I \to I$ is a function whose zero c is to be approximated using Newton's method. The Newton function is $N_f(x) = x - f(x)/f'(x)$, where we are assuming $f'(c) \neq 0$. Notice that since $f(c) = 0$, $N_f(c) = c$, i.e., c is a fixed point of N_f. Consider $N'_f(c)$:

$$N'_f(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

so that

$$N'_f(c) = \frac{f(c)f''(c)}{[f'(c)]^2} = 0,$$

since $f(c) = 0$.

It follows that $|N_f'(c)| = 0 < 1$, so that c is an attracting fixed point for N_f, and in particular

$$\lim_{n\to\infty} N_f^n(x_0) = c,$$

provided x_0, the first approximation to c, is sufficiently close to c.

Definition 1.4.9 A fixed point c for $f(x)$ is called a *super-attracting fixed-point* if $f'(c) = 0$. This gives a very fast convergence to the fixed-point for points nearby.

Remark 1.4.10 Suppose that $f'(c) = 0$, then $N_f(x)$ is not defined at $x = c$, since the quotient $f(x)/f'(x)$ is not defined there. Suppose we can write $f(x) = (x - c)^k h(x)$ where $h(c) \neq 0$ and $k \in \mathbb{Z}^+$ (for example if $f(x)$ is a polynomial with a multiple root), then we have

$$\frac{f(x)}{f'(x)} = \frac{(x-c)^k h(x)}{k(x-c)^{k-1}h(x) + (x-c)^k h'(x)} = \frac{(x-c)h(x)}{kh(x) + (x-c)h'(x)} = 0,$$

when $x = c$, so that $N_f(x) = c$ has a removable discontinuity at $x = c$ (removed by setting $N_f(c) = c$). Then we can find $N_f'(x)$ and show that $N_f'(x)$ has a removable discontinuity at $x = c$, which can again be removed by setting $N_f'(c) = (k-1)/k$, so that $|N_f'(c)| < 1$ (see the exercises).

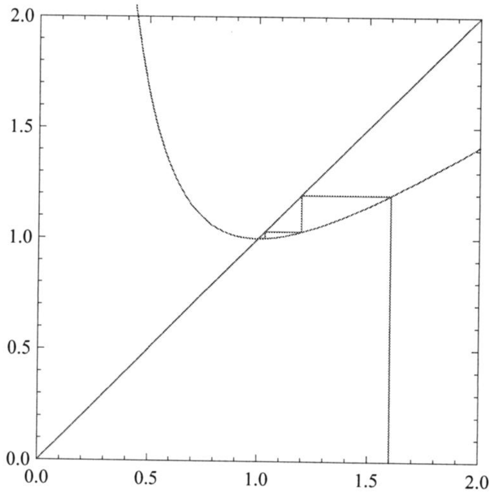
We summarize the above with a theorem:

Theorem 1.4.11 *For a differentiable function $f : I \to I$, the zero c of $f(x)$ is a super-attracting fixed point of the Newton function N_f, if and only if $f'(c) \neq 0$.*

Example 1.4.12 Suppose $f(x) = x^3 - 1$, then $f(1) = 0$ and

$$N_f(x) = x - f(x)/f'(x) = x - \left(\frac{x^3-1}{3x^2}\right) = \frac{2x}{3} + \frac{1}{3x^2},$$

so $N_f(1) = 1$ and $N_f'(x) = \frac{2}{3} - \frac{2}{3x^3}$, so that $N_f'(1) = 0$. We see from graphical iteration (next section), very fast convergence to the fixed point of N_f.



Very fast convergence to the fixed point.

Exercises 1.4

1. Find the fixed points and determine their stability for the function $f(x) = 4 - \frac{3}{x}$.

2. For the family of quadratic maps $Q_c(x) = x^2 + c$, $x \in [0, 1]$, use the Mathematica program `webPlot` below to give graphical iteration for the values shown (use 20 iterations):

(i) $c = 1/2$, starting point $x_0 = 1$, (ii) $c = 1/4$, starting point $x_0 = .1$, (iii) $c = 1/8$, starting point $x_0 = .7$.

Mathematica Program: Use as in:

```
webPlot[Cos[x], {x, 0, Pi}, {.1, 20}, AspectRatio->Automatic]
```

```
webPlot[g_, {x_, xmin_, xmax_}, {a1_, n_}, opts_] :=
 Module[{seq, r, pts, web, graph}, r[t_] := N[g /. x -> t];
  seq = NestList[r, a1, n];      pts = Flatten[
Table[{{seq[[i]],seq[[i + 1]]},{seq[[i + 1]],seq[[i + 1]]}},{i, 1, n}], 1];
  web = Graphics[{Hue[0], Line[PrependTo[pts, {seq[[1]], 0}]]}];
  graph = Plot[{x, r[x]}, {x, xmin, xmax},
    DisplayFunction -> Identity]; Print["last iterate =", Last[seq]];
  Show[web, graph, opts, Frame -> True,
```

```
PlotRange -> {{xmin, xmax}, {xmin, xmax}}]]
```

3. Let $S_\mu(x) = \mu \sin(x)$, $0 \leq \mu \leq 2\pi$, $0 \leq \mu \leq \pi$ and $C_\mu(x) = \mu \cos(x)$, $-\pi \leq x \leq \pi$ and $0 \leq \mu \leq \pi$.

(a) Show that S_μ has a super-attracting fixed point at $x = \pi/2$, when $\mu = \pi/2$.

(b) Find the corresponding values for C_μ having a super-attracting fixed point.

4. Show that the map $f(x) = \dfrac{2}{x+1}$ has no periodic points of period $n > 1$ (Hint: Use the closed formula in Exercise 1.1.6).

5. Let N_f be the Newton function of the map $f(x) = x^2 + 1$. Clearly there are no fixed points of the the Newton function as there are no zeros of f. Show that there are points c where $N_f^2(c) = c$ (called *period 2-points* of N_f).

6. (a) Suppose that $f(c) = f'(c) = 0$ and $f''(c) \neq 0$. If $f''(x)$ is continuous at $x = c$, show that the Newton function $N_f(x)$ has a removable discontinuity at $x = c$ (hint: apply L'Hopital's rule to N_f at $x = c$).

(b) If in addition, $f'''(x)$ is continuous at $x = c$ with $f'''(c) \neq 0$, show that $N_f'(c) = 1/2$, so that $x = c$ is not a super-attracting fixed point in this case.

(c) Check the above for the function $f(x) = x^3 - x^2$ with $c = 0$.

7. Continue the argument in Remark 1.4.11 (generalizing the last exercise), to show that the derivative of the Newton function N_f has a removable discontinuity at $x = c$, which can be removed by setting $N_f'(c) = (k-1)/k$.

8. Let f be a twice differentiable function with $f(c) = 0$. Show that if we find the Newton function of $g(x) = f(x)/f'(x)$, then $x = c$ will be a super-attracting fixed point for N_g, even if $f'(c) = 0$ (this is called Halley's method).

1.5 Non-hyperbolic Fixed Points.

$|f'(c)| = 1$

Example 1.5.1 We have seen that it $f : X \to X$ (where usually X is some subinterval of $\mathbb{R}$) and $a \in X$ with $f(a) = a$, $|f'(a)| < 1$, with suitable differentiability conditions, then a is a stable fixed point for f. This term is used because points close to a will approach a under iteration. If we look at a function like $f(x) = \sin x$, we see that $c = 0$ is a fixed point but in this case $|f'(0)| = 1$, so it is a non-hyperbolic fixed point. However, graphical iteration suggests that the basin of attraction of f is all of $\mathbb{R}$, so $c = 0$ is a stable fixed point. Before considering non-hyperbolic fixed points in more detail, let us prove the last statement analytically:

Proposition 1.5.2 *The fixed point $c = 0$ of $f(x) = \sin x$ is globally attracting .*

Proof. First notice that $c = 0$ is the only fixed point of $\sin x$. This is clear for if $\sin x = x$ then we cannot have $|x| > 1$ since $|\sin x| \leq 1$ for all x. If $0 < x \leq 1$, the Mean Value Theorem implies there exists $c \in (0, x)$ with

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin x}{x},$$

so that

$$\sin x = x \cos c < x,$$

since $|\cos c| < 1$ for $c \in (0, 1)$. Similarly if $-1 \leq x < 0$.

To show that $f(x)$ is globally attracting, let $x \in \mathbb{R}$. We may assume that $-1 \leq x \leq 1$, since this will be the case after the first iteration.

Suppose that $0 < x \leq 1$, then we note that $0 < f'(x) < 1$ on this interval. By the Mean Value Theorem, there exists $c \in (0, x)$ with

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad \text{or} \quad 0 < f(x) = f'(c)x < x.$$

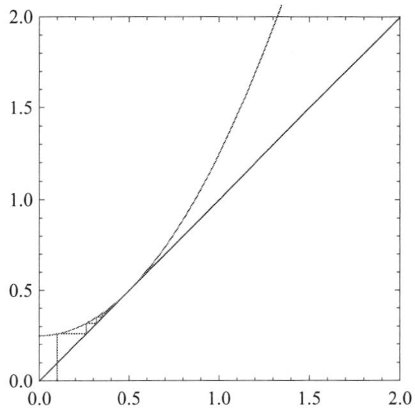
Continuing in this way, we see that

$$0 < f^n(x) < f^{n-1}(x) < \ldots < f(x) < x,$$

so we have a decreasing sequence $x_n = f^n(x)$ bounded below by 0. It follows that this sequence converges. But Proposition 1.4.5 implies that if this sequence converges, it must converge to a fixed point. $c = 0$ being the only fixed point gives $f^n(x) \to 0$ as $n \to \infty$. A similar argument can be used for $-1 \leq x < 0$. $\square$

Example 1.5.3 It is possible for the fixed point to be unstable, but to have a one-sided stability (to be *semi-stable*). For example, consider $f(x) = x^2 + 1/4$ which has

the single (non-hyperbolic) fixed point $c = 1/2$. This fixed point is stable from the left, but unstable on the right.



In the following we give some criteria for non-hyperbolic fixed points to be asymptotically stable/unstable etc. It also gives a criteria for semi-stability.

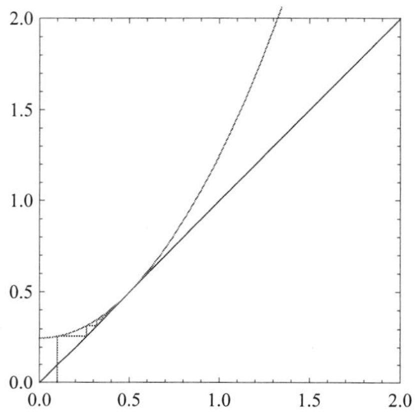
Theorem 1.5.4 *Let c be a non-hyperbolic fixed point of $f(x)$ with $f'(c) = 1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at $x = c$, then:*

(i) *if $f''(c) \neq 0$, the c is semi-stable,*

(ii) *if $f''(c) = 0$ and $f'''(c) > 0$, then c is unstable,*

(iii) *If $f''(c) = 0$ and $f'''(c) < 0$, then c is asymptotically stable.*

Proof. (i) If $f'(c) = 1$ then $f(x)$ is tangential to $y = x$ at $x = c$. Suppose that $f''(c) > 0$, then $f(x)$ is concave up at $x = c$ and the picture must look like the following:



We see this gives stability on the left and instability on the right.

More formally, since the derivatives are continuous, and $f''(c) > 0$, this will be true in some small interval $(c - \delta, c + \delta)$ surrounding c. In particular, the derivative function $f'(x)$ must be increasing on that interval, so that since $f'(c) = 1$ then

$$f'(x) < 1 \quad \text{for all} \quad x \in (c - \delta, c), \quad \text{and} \quad f'(x) > 1 \quad \text{for all} \quad x \in (c, c + \delta),$$

for some $\delta > 0$. Also, from the continuity of $f'(x)$, we can assume that $f'(x) > 0$ in this interval.

Now by the Mean Value Theorem applied to the interval $[x, c] \subset (c - \delta, c]$, there exists $q \in (x, c)$ with

$$f'(q) = \frac{f(x) - f(c)}{x - c},$$

Now since $0 < f'(q) < 1$ and $c > x$, we have

$$0 < \frac{f(x) - f(c)}{x - c} < 1,$$

or

$$x < f(x) < c.$$

Repeating this argument, we see that the sequence $f^n(x)$ is increasing and bounded above by c, so must converge to a fixed point. There can be no other fixed point (say $d \neq c$), in this interval as the Mean Value Theorem would give $f'(q_1) = 1$ for some $q_1 \in (x, c)$, a contradiction. Consequently we see that c is stable on the left.

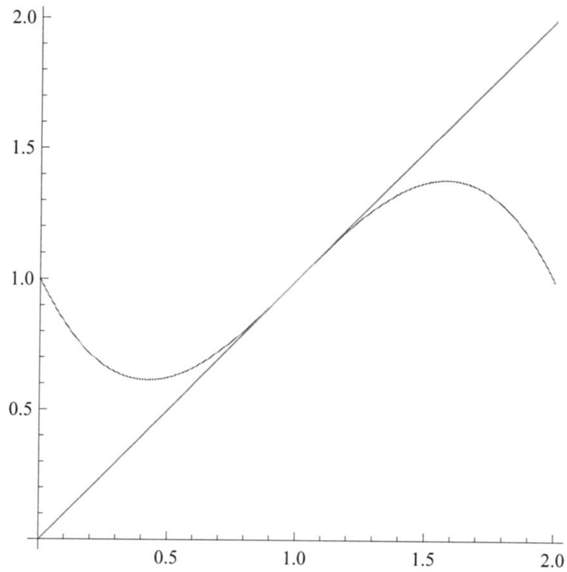
On the other hand, if $[c, x] \subset (c, c + \delta)$, then applying the Mean Value Theorem as above gives

$$f'(q) = \frac{f(x) - f(c)}{x - c} > 1, \quad \text{so} \quad f(x) > x > c,$$

since $x - c > 0$. This tells us that the point moves away from c under iteration, so the fixed point is unstable on the right. Similar considerations can be used when $f''(c) < 0$ and the graph is concave down at $x = c$.

(ii) is similar to (iii), so is omitted.

(iii) In this case $f'''(c) < 0$, $f''(c) = 0$ and $f'(c) = 1$. We will show that we have a point of inflection at $x = c$ as in the following picture:



By the second derivative test, $f'(x)$ has a local maximum at $x = c$ (the continuous function $f'(x)$ is concave down). It follows that

$$f'(x) < 1 \quad \text{for all} \quad x \in (c - \delta, c + \delta),\ x \neq c,$$

for some $\delta > 0$ ($f''(x) > 0$ for $x \in (c-\delta, c)$, so $f'(x)$ is increasing there, and $f''(x) < 0$ for $x \in (c, c+\delta)$, so $f'(x)$ is decreasing there). In particular $f'(x) \neq 1$ if $x \neq c$.

Now use an argument similar to that of (i) above to deduce the result. □

Example 1.5.5 Returning to the function $f(x) = \sin x$, we see that $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = -1$, so the conditions of Theorem 1.5.4 (iii) are satisfied and we conclude that $x = 0$ is an asymptotically stable fixed point.

If $f(x) = \tan x$, then $f'(0) = 1$ and we can check that Theorem 1.5.4 (ii) holds so that the fixed point $x = 0$ is unstable. For $f(x) = x^2 + 1/4$, with $f(1/2) = 1/2$, $f'(1/2) = 1$, we can apply Theorem 1.5.4 (i).

How do we treat the case where $f'(c) = -1$ at the fixed point? We use:

Definition 1.5.6 The *Schwarzian derivative* $Sf(x)$ of $f(x)$ is the function

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2,$$

so that

$$Sf(x) = -f'''(x) - \frac{3}{2}[f''(x)]^2, \quad \text{when} \quad f'(x) = -1.$$

Theorem 1.5.7 *Suppose that c is a fixed point for $f(x)$ and $f'(c) = -1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at $x = c$ then:*

(i) *if $Sf(c) < 0$, then $x = c$ is an asymptotically stable fixed point,*

(ii) *if $Sf(c) > 0$, then $x = c$ is an unstable fixed point.*

Proof. (i) Set $g(x) = f^2(x)$, then $g(c) = c$ and we see that if c is asymptotically stable with respect to g, then it is asymptotically stable with respect to f. Now

$$g'(x) = \frac{d}{dx}(f(f(x))) = f'(f(x)) \cdot f'(x),$$

so that $g'(c) = f'(c) \cdot f'(c) = (-1)(-1) = 1$.

The idea is to apply Theorem 1.5.4 (iii) to the function $g(x)$. Now

$$g''(x) = f'(f(x)) \cdot f''(x) + f''(f(x)) \cdot [f'(x)]^2,$$

thus

$$g''(c) = f'(c)f''(c) + f''(c)[f'(c)]^2 = 0, \quad \text{since} \quad f'(c) = -1.$$

Also

$$g'''(x) = f''(f(x)) \cdot f'(x) \cdot f''(x) + f'(f(x)) \cdot f'''(x) + f'''(f(x))[f'(x)]^3 + f''(f(x)) \cdot 2f'(x)f''(x).$$

Therefore

$$\begin{aligned} g'''(c) &= [f''(c)]^2(-1) - f'''(c) - f'''(c) + 2f''(c)(-1)f''(c) \\ &= -2f'''(c) - 3[f''(c)]^2 \\ &= 2Sf(c) < 0, \end{aligned}$$

and the result follows from Theorem 1.5.4 (iii).

(ii) This now follows from Theorem 1.5.4 (ii) □

Remark 1.5.8 The above proof shows how the Schwarzian derivative arises from differentiating $g = f \circ f = f^2$. In the case where $f'(c) = -1$, it follows that

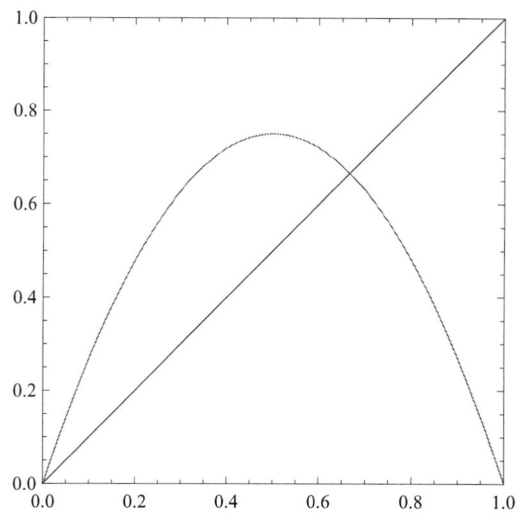
$$g''(c) = 0 \quad \text{and} \quad Sf(c) = \frac{1}{2}g'''(c).$$

Example 1.5.9 For the logistic map $L_\mu(x) = \mu x(1 - x)$ we have

$$L'_\mu(x) = \mu - 2\mu x, \quad L''_\mu(x) = -2\mu \quad \text{and} \quad L'''_\mu(x) = 0.$$

When $\mu = 1$, $x = 0$ is the only fixed point and Theorem 1.5.4 (i) shows that $x = 0$ is semi-stable (attracting on the right). However, we regard this as a stable fixed point for L_μ defined on the interval $[0, 1]$, since points to the left of 0 are not in the domain of L_μ.

When $\mu = 3$, $c = 2/3$ is fixed and $L'_\mu(2/3) = -1$ giving a non-hyperbolic fixed point. However, $Sf(2/3) = 0 - \frac{3}{2}[6]^2 < 0$ (negative Schwarzian derivative), so by Theorem 1.5.7 (i), $x = 2/3$ is asymptotically stable.



Exercises 1.5

1. Show that $f(x) = -2x^3+2x^2+x$ has two non-hyperbolic fixed points and determine their stability.

2. For the family of quadratic maps $Q_c(x) = x^2 + c$, $x \in \mathbb{R}$, use the Theorems of Section 1.5 to determine the stability of the fixed points for all possible values of c. Find any values of c so that Q_c has a non-hyperbolic fixed point, and determine their stability.

3. Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable or unstable:

$$\text{(i) } f(x) = \frac{x^3}{2} + \frac{x}{2}, \quad \text{(ii) } f(x) = \arctan x, \text{ (iii) } f(x) = x^3 + x^2 + x,$$

$$\text{(iv) } f(x) = x^3 - x^2 + x, \quad \text{(v) } f(x) = \begin{cases} 0.8x & ;\text{if} \quad x \le 1/2 \\ 0.8(1-x) & ;\text{if} \quad x > 1/2 \end{cases}$$

4. Let $f(x) = ax^2 + bx + c$, $a \neq 0$, and p a fixed point of f. Prove the following:

(i) If $f'(p) = 1$, then p is semistable.

(ii) If $f'(p) = -1$, then p is asymptotically stable.

5. If $f(x) = \dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$ is the linear fractional transformation, show that its Schwarzian derivative is $Sf(x) = 0$ for all x in its domain.

6. If $Sf(x)$ is the Schwarzian derivative of $f(x)$, a C^3 function and $F(x) = \dfrac{f''(x)}{f'(x)}$, show that $Sf(x) = F'(x) - (F(x))^2/2$.

7. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, find the periodic points of f and show that for $x \neq 0$ they are non-hyperbolic. Show that $x = 0$ is not an isolated fixed point (i.e., there are other fixed points arbitrarily close to 0). Is $x = 0$ a stable, attracting or repelling fixed point?

8. ([37]) Let N_f be the Newton function of a four times continuously differentiable function f. If $f(\alpha) = 0$, show that $N_f'''(\alpha) = 2Sf(\alpha)$, where Sf is the Schwarzian derivative of f.

9. (a) Use the Intermediate Value Theorem to show that $f(x) = \cos(x)$ has a fixed point c in the interval $[0, \pi/2]$. We can show experimentally that this fixed point is approximately $c = .739085\ldots$, for example by iterating any $x_0 \in \mathbb{R}$.

(b) Show that the basin of attraction of c is all of $\mathbb{R}$ (Hint: You may assume that $x \in [-1, 1]$ - why? Now use the Mean Value Theorem to show that $|f(x) - c| < \lambda |x - c|$ for some $0 < \lambda < 1$).

(c) Does $f(x)$ have any eventual fixed points?

(d) Can $f(x)$ have any points p with $f^2(p) = p$ other than c?

[illegible]
ص 110

<u>Example</u> (Symbolic Dynamics $\sum_2^+$)

Consider $\sum_2^+ = \{ x = \{x_n\}_{n=0}^{\infty} : x_n = 0 \text{ or } 1 \}$

امثلة عن العناصر
$\{100111\cdots\}$
$\{0\bar{1}\} = \{01111\ldots\}$
$\{\bar{0}\}, \{\bar{1}\}$

define a map (Shift (left) map)

$\sigma : \sum_2^+ \longrightarrow \sum_2^+$ by

$\sigma(\{x_0 x_1 x_2 x_3 \cdots\}) = \{x_1 x_2 x_3 \cdots\}$

Find ① All fixed points of σ
② All periodic points
③ all eventually f. Pts.
④ all eventuall per. Pts.

Chapter 2, Bifurcations and the Logistic Family

2.1 The Basin of Attraction

In this chapter we examine the basin of attraction of the logistic maps, i.e., for a given μ and fixed point $x = c$ we look for the set of those $x \in [0, 1]$ which converge to c under iteration by $L_\mu(x) = \mu x(1 - x)$.

Definition 2.1.1 The *basin of attraction* $B_f(c)$ of a fixed point c of $f(x)$ is the set of all x for which the sequence $x_n = f^n(x)$ converges to c:

$$B_f(c) = \{x \in X : f^n(x) \to c, \quad \text{as } n \to \infty\}. \quad = B_c$$

The *immediate basin of attraction* of f is the largest interval containing c, contained in the basin of attraction of c. We first show that this is always an open interval when c is an attracting fixed point.

Proposition 2.1.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function having an attracting fixed point p. The immediate basin of attraction of p is an open interval.*

W_p

Proof. Since p is an attracting fixed point there is $\epsilon > 0$ such that for all $x \in I_\epsilon = (p - \epsilon, p + \epsilon)$, $f^n(x) \to p$ as $n \to \infty$. Denote by J the largest interval containing p for which $f^n(x) \to p$ for $x \in J$, as $n \to \infty$.

Suppose that $J = [a, b]$, a closed interval, then there exists $r \in \mathbb{Z}^+$ with $f^r(a) \in I_\epsilon$. Now f^r is also a continuous function, so points close to a will also get mapped into I_ϵ, leading to a contradiction.

More specifically, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f^r(x) - f^r(a)| < \eta$, where $\eta = \min\{|f^r(a) - (p - \epsilon)|, |(p + \epsilon) - f^r(a)|\}$. Thus there are points $x < a$ close to a for which $f^r(x) \in I_\epsilon$, so $f^{rn}(x) \to p$ as $n \to \infty$, a contradiction. We conclude that $a \notin J$ (and similarly for b), so J is an open interval. □

Example 2.1.3 1. If $f(x) = x^2$, the fixed points are $c = 0$ and $c = 1$, both hyperbolic, the first being attracting and the second repelling. Clearly $B_f(0) = (-1, 1)$ and $B_f(1) = \{-1, 1\}$. We sometimes regard $c = \infty$ as an attractive fixed point of f, so that $B_f(\infty) = [-\infty, -1) \cup (1, \infty]$.

2. If $f : I \to I$ is a continuous function on a closed interval $I = [a, b]$ having an attracting fixed point $p \in (a, b)$, then we cannot exclude the possibility that the basin of attraction includes either a or b, or both. On the other hand, if $x = a$ is the

$x \in \mathbb{R}$, $a \geq 0$, $\boxed{f(x) = x^2 + a}$ نبدأ بالمثال

$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a}$

$P_+ = \frac{1}{2} + \frac{1}{2}\sqrt{1-4a}$, $P_- = \frac{1}{2} - \frac{1}{2}\sqrt{1-4a}$

so for $\mu \leq \frac{1}{4}$ we have a f.p.

attracting fixed point, the bsain of attraction may be a set of the form $[a, c)$ for some $c \in (a, b]$. Such a set can be regarded as being open as a subset of $[a, b]$.

Example

2.2 The Logistic Family

The logistic maps $L_\mu(x) = \mu x(1 - x)$ are functions of two real variables μ and x. We usually restrict x to the interval $[0, 1]$ and consider $\mu \in (0, 4]$. μ is a parameter which we allow to vary, but then we study the function L_μ for specific fixed values of μ. As the parameter μ is varied, we see a corresponding change in the nature of the function L_μ. This is what is called *bifurcation*. For example, for $0 < \mu \leq 1$, L_μ has exactly one fixed point in $[0, 1]$, $c = 0$, which is attracting. As μ increases beyond 1, a new fixed point $c = 1 - 1/\mu$, is created in $[0, 1]$, so now L_μ has two fixed points. $c = 0$ is now repelling and $c = 1 - 1/\mu$ is attracting (for $1 < \mu \leq 3$). At $\mu = 3$ the nature of these fixed points changes again as we shall see. In this section we determine the basin of attraction of these fixed points as μ increases from 0 to 3. We see that the "dynamics" (long term behavior) of L_μ is quite uncomplicated for this range of values of μ.

The function $L_\mu(x) = \mu x(1 - x)$, $0 \leq x \leq 1$ has a maximum value of $\mu/4$ when $x = 1/2$. Consequently, for $0 < \mu \leq 4$, L_μ maps the unit interval $[0, 1]$ into itself. We shall consider later what happens when $\mu > 4$. We start by showing that the basin of attraction of L_μ for $0 < \mu \leq 1$ is all of the domain of L_μ, namely $[0, 1]$. We say that 0 is a *global attractor* in this case.

Theorem 2.2.1 *Let $L_\mu(x) = \mu x(1-x)$, $0 \leq x \leq 1$ be the logistic map. For $0 < \mu \leq 1$, $B_{L_\mu}(0) = [0, 1]$ and for $1 < \mu \leq 3$, $B_{L_\mu}(1 - 1/\mu) = (0, 1)$.*

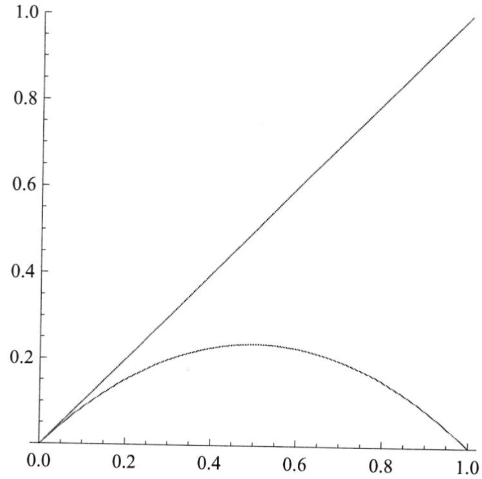
We split the proof into a number of different cases:

Case 2.2.2 $\mathbf{0 < \mu \leq 1}$. → page 40

Example: Consider the map $g : [-2, 4] \longrightarrow [-2, 4]$ that defined as

$$g(x) = \begin{cases} x^2 & \text{if } -2 \leq x \leq 1 \\ 3\sqrt{x} - 2 & \text{if } 1 < x \leq 4 \end{cases}$$

this map has three fixed points $c_1 = 0$, $c_2 = 1$ and $c_3 = 4$, and the basin of attraction of them are $B_{c_1} = (-1, 1)$, $B_{c_3} = [-2, -1) \cup (1, 4]$ and $B_{c_2} = \{-1, 1\}$.



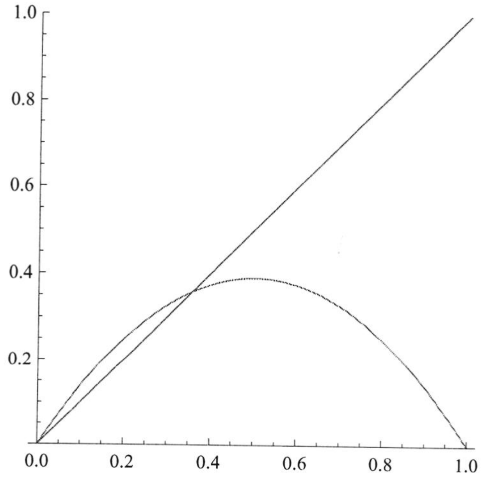
For $0 < \mu < 1$, the only fixed point is 0.

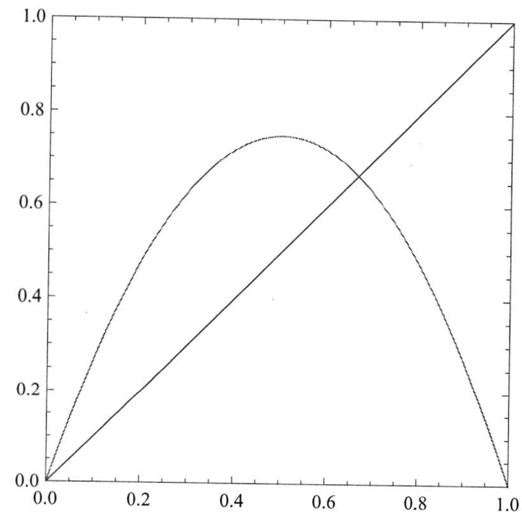
We have seen that for $\mu \in (0,1)$, L_μ has only the one fixed point $x = 0$ in $[0,1]$ (the other fixed point is $1 - 1/\mu \leq 0$). For $\mu < 1$ this fixed point is asymptotically stable, but in any case

$$0 < \mu \leq 1, \ \ 0 < 1 - x < 1 \Rightarrow 0 < \mu(1-x) < 1$$
$$\Rightarrow 0 < L_\mu(x) = \mu x(1-x) < x, \ \ x \in (0,1],$$

and in a similar way, $L_\mu^2(x) < L_\mu(x)$ etc., so the sequence $L_\mu^n(x)$ is decreasing, bounded below by 0, and hence must converge to the only fixed point, namely 0. It follows that the basin of attraction is $B_{L_\mu}(0) = [0,1]$ $(L_\mu(1) = 0)$.

Case 2.2.3 $\mathbf{1 < \mu \leq 3}$.





$1 - 1/\mu < 1/2$ for $1 < \mu < 2$. $\quad$ $1 - 1/\mu \geq 1/2$ for $2 \leq \mu \leq 3$.

We have seen that for $\mu > 1$ the fixed point 0 is repelling, but a new fixed point $c = 1 - 1/\mu$ has been "born" which is attracting (for $1 < \mu \leq 3$), so by Proposition 2.1.2 there is a largest open interval $I = (a, b)$ containing the fixed point for which $x \in I$ implies $L_\mu^n(x) \to c$ as $n \to \infty$. If the basin of attraction of c is $B_\mu(c)$, then $0, 1 \notin B_\mu(c)$ because $L_\mu(0) = 0$ and $L_\mu(1) = 0$, so $B_\mu(c) \neq [0, 1]$. Furthermore, clearly $a, b \notin B_\mu(c)$.

From the Intermediate Value Theorem, $L_\mu(a, b)$ is an interval which must be contained in (a, b), for if $x \in (a, b)$, $L_\mu^n(L_\mu(x)) \to c$ as $n \to \infty$. Let x_n be a sequence in (a, b) with $\lim_{n\to\infty} x_n = a$, then by the continuity of L_μ, $\lim_{n\to\infty} L_\mu(x_n) = L_\mu(a)$. Since $x_n \in (a, b)$ for every $n \in \mathbb{Z}^+$, we have $L_\mu(x_n) \in (a, b)$ for all $n \in \mathbb{Z}^+$. Since $L_\mu(a) \notin (a, b)$, the only way this is possible is if $L_\mu(a) = a$ or $L_\mu(a) = b$, and similarly for b. This is only possible if a and b are fixed points, are eventual fixed points or $L_\mu(a) = b, L_\mu(b) = a$ (we call $\{a, b\}$ a 2-cycle - see Section 2.3). We show that the latter case cannot happen, and this leads to the conclusion that $a = 0$ and $b = 1$, since there are no other fixed or eventual fixed points in $[0, 1]$ that can satisfy these condition. Consequently, we must have $B_\mu(1 - 1/\mu) = (0, 1)$.

Now we can check that

$$L_\mu^2(x) - x = \mu^2 x(1 - (\mu+1)x + 2\mu x^2 - \mu x^3) - x = x(\mu^2 x^2 - \mu(\mu+1)x + \mu + 1)(\mu x - \mu + 1),$$

and a and b must satisfy this equation. We can disregard the linear factors as they give the two fixed points. The discriminant of the quadratic factor is $\mu^2(\mu + 1)^2 - 4\mu^2(\mu+1) = \mu^2(\mu+1)(\mu-3) < 0$ for $1 < \mu < 3$, so there is no 2-cycle when $1 < \mu < 3$. When $\mu = 3$, the discriminant is zero, the fixed point is $c = 2/3$ and the quadratic factor gives rise to no additional roots, so again there is no 2-cycle. $\square$

Remark 2.2.4 Note that when $\mu = 2$, $x = 1/2$ is a super attracting fixed point (since $L_2'(1/2) = 0$). As we saw above the basin of attraction will be $(0, 1)$.

Exercises 2.2

1. If $L_\mu(x) = \mu x(1 - x)$ is the logistic map, show that $x = 1/2$ is the only turning point of $L^2(x)$ for $0 < \mu \leq 2$, but when $\mu > 2$, two new turning points are created.

Use this to show that for $2 < \mu < 3$, the interval $[1/\mu, 1 - 1/\mu]$ is mapped by L_μ^2 onto the interval $[1/2, 1 - 1/\mu]$.

2.3 Periodic Points

Points with finite orbits are of importance in the study of dynamical systems and their long-term behavior:

Definition 2.3.1 Let $f : X \to X$ be a function with $c \in X$.

(i) c is a *periodic point* of $f(x)$ with period $r \in \mathbb{Z}^+$ if $f^r(c) = c$ and $f^k(c) \neq c$ for $0 < k < r$ (in particular, c is a fixed point of f^r). We call r the *period* of c and the set $O(c) = \{c, f(c), f^2(c), \ldots, f^{r-1}(c)\}$ is an r*-cycle.* We write

$$\mathrm{Per}_r(f) = \{x \in X : f^r(x) = x\},$$

so that $\mathrm{Fix}(f) \subseteq \mathrm{Per}_n(f)$, $n = 1, 2, \ldots$, since the points in $\mathrm{Per}_n(f)$ may not be of period n, but of some lesser period.

(ii) c is *eventually periodic* for f if there exists $m \in \mathbb{Z}^+$ such that $f^m(c)$ is a periodic point of f (we assume that c is not a periodic point).

(iii) c is *stable* (respectively *asymptotically stable, unstable* etc.) if it is a stable fixed point of f^r.

The following criteria for stability now immediately follows from Theorem 1.3.3:

Theorem 2.3.2 *Suppose that c is a point of period r for f and that $f'(x)$ is continuous at $x = c$. If $c_i = f^i(c)$, $i = 0, 1, \ldots, r-1$ then:*

(i) *c is asymptotically stable if*

$$|f'(c_0) \cdot f'(c_1) \cdot f'(c_2) \cdots f'(c_{r-1})| < 1.$$

(ii) *c is unstable if*

$$|f'(c_0) \cdot f'(c_1) \cdot f'(c_2) \cdots f'(c_{r-1})| > 1.$$

Proof. Let us look at the case where $r = 3$ as this is typical;

$$O(c) = \{c, f(c), f^2(c)\} = \{c_0, c_1, c_2\}.$$

Then

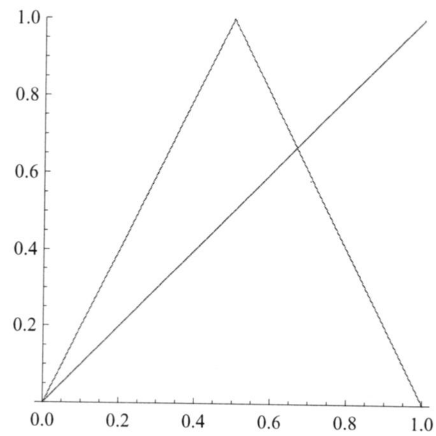
$$\frac{d}{dx}(f^3(x)) = \frac{d}{dx}(f(f^2(x))) = f'(f^2(x))(f^2(x))' = f'(f^2(x))f'(f(x))f'(x)$$

$$= f'(c_2)f'(c_1)f'(c_0), \quad \text{when} \quad x = c.$$

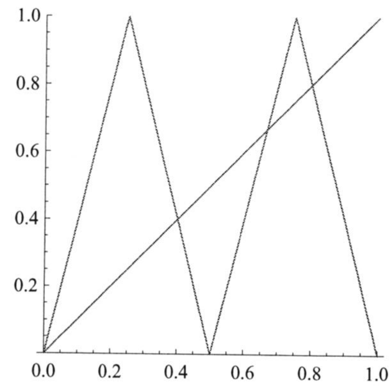
The result now follows from 1.3.3. □

Example 2.3.3 If we look at the graph of the tent map $T(x)$, we see it has two fixed points, $c = 0$ and $c = 2/3$. If we graph T^3 it has eight fixed points: These arise from two 3-cycles: $\{2/7, 4/7, 6/7\}$ and $\{2/9, 4/9, 8/9\}$, together with the two fixed points, so that

$$\text{Per}_3(T) = \{0, 2/3, 2/7, 4/7, 6/7, 2/9, 4/9, 8/9\}.$$



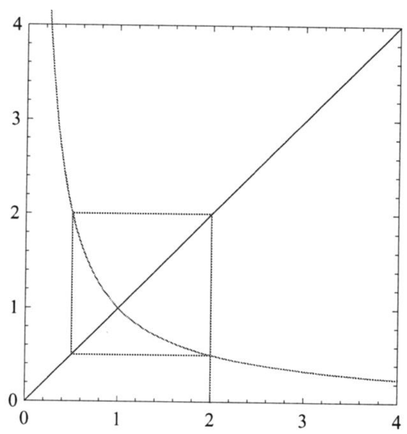
The graph of T^2 shows that T^2 has four fixed points coming from a 2-cycle $\{2/5, 4/5\}$ and the two fixed points. Since $|T'(x)| = 2$ and $|(T^2)'(x)| = |T'(x)||T'(Tx)| = 4$, etc. (except at points of non-differentiability), all these periodic points will be unstable.



Example 2.3.4 Let $f(x) = 1/x$, $x \neq 0, x \neq \pm 1$. Note that $f^2(x) = x$ and $f(x) \neq x$ for all such x, giving rise to the 2-cycle $\{x, 1/x\}$. In this case

$$|f'(x)f'(1/x)| = |-1/x^2(-x^2)| = 1,$$

so the theorem is inconclusive. However we see that the periodic points are stable but neither attracting nor repelling.



Example 2.3.5 Consider the quadratic function $f(x) = x^2 - 2$. We have seen that to find the fixed points we solve $f(x) = x$, or $x^2 - 2 = x$, $x^2 - x - 2 = (x-2)(x+1) = 0$, so $x = 2$ or $x = -1$.

To find the period 2-points we solve $f^2(x) = x$ or $f^2(x) - x = 0$. This is simplified when we realize that the fixed points must be solutions of this equation, so that $(x-2)(x+1)$ is a factor. We can then check that

$$f^2(x) - x = x^4 - 4x^2 - x + 2 = (x-2)(x+1)(x^2 + x - 1).$$

Solving the quadratic gives

$$x = \frac{-1 \pm \sqrt{5}}{2},$$

so that $\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\}$ is a 2-cycle. In general, finding periodic points of quadratics can be complicated. If $f(x)$ is a quadratic, $f^n(x) - x$ is a polynomial of degree 2^n.

To check the stability, we calculate

$$|f'((-1-\sqrt{5})/2)f'((-1+\sqrt{5})/2)| = |(-1-\sqrt{5})(-1+\sqrt{5})| = |1-5| = 4 > 1,$$

giving an unstable 2-cycle.

Remark 2.3.6 1. As before, periodic points can be stable but not attracting (as above with $f(x) = 1/x$ at $x \neq 1$). They can also be attracting but not stable as in Example 1.4.7).

2. Functions such as $f(x) = \sin x$ can have no period 2 points or points of a higher period since this would contradict the basin of attraction of $x = 0$ being all of $\mathbb{R}$. Similarly, the logistic map L_μ, $0 < \mu \leq 3$ cannot have period n-points for $n > 1$.

Exercises 2.3

1. For each of the following functions, $c = 0$ lies on a periodic cycle. Classify this cycle as attracting, repelling or neutral (non-hyperbolic). Say if it is super-attracting:

$$\text{(i)}\ f(x) = \frac{\pi}{2}\cos x, \quad \text{(ii)}\ g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

2. (a) Show that $C_\mu(x) = \mu\cos(x)$ has a super-attracting 3-cycle $\{0, \lambda, \pi/2\}$ where $\mu = \lambda$ and λ satisfies the equation $\lambda\cos(\lambda) = \pi/2$.

(b) Give similar conditions for $S_\mu(x) = \mu\sin(x)$ to have (i) a super-attracting 2-cycle, (ii) a super-attracting 3-cycle.

(c) Explain why for families of maps, say F_μ, one member of a super-attracting n-cycle is a super-attracting fixed point (for a different value of μ).

3. Find the fixed points and the period two points of following maps (if any) and determine the stability of the 2-cycle:

(i) $f(x) = \frac{x^2}{2} - x + \frac{1}{2}$, (ii) $f(x) = a - \frac{b}{x}$ $(b \neq 0)$, (iii) $f(x) = \frac{1-x}{3x+1}$, (iv) $f(x) = |x-1|$.

4. Let $Q_c(x) = x^2 + c$. Show that for $c < -3/4$, Q_c has a 2-cycle and find it explicitly. For what values of c is the 2-cycle attracting?

5. Let $f(x) = ax^2 + bx + c$, $a \neq 0$. Let $\{x_0, x_1\}$ be a 2-cycle for $f(x)$,

(a) If $f'(x_0)f'(x_1) = -1$, prove that the 2-cycle is asymptotically stable.
(b) If $f'(x_0)f'(x_1) = 1$, prove that the 2-cycle is asymptotically stable.

6. Let $f(x) = ax^3 - bx + 1$, $a \neq 0$. If $\{0, 1\}$ is a 2-cycle for $f(x)$, give conditions on a and b under which it is asymptotically stable.

7. Let $f(x)$ be a polynomial with $f(c) = c$.

(i) If $f'(c) = 1$, show that $(x-c)^2$ is a factor of $g(x) = f(x) - x$.

(ii) If $f'(c) = 1$, or $f'(c) = -1$, show that $(x-c)^2$ is a factor of $h(x) = f^2(x) - x$ (i.e., if $f(x)$ has a non-hyperbolic fixed point c, then c is a repeated root of $f^2(x) - x$). Hint: Recall that a polynomial $p(x)$ has $(x-c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.

(iii) Show in the case that $f'(c) = -1$ we actually have that $(x-c)^3$ is a factor of $h(x) = f^2(x) - x$.

(iv) Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$, of the logistic map $L_3(x) = 3x(1-x)$.

(v) Check that (i), (ii) and (iii) hold for the (non-hyperbolic) fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable everywhere. If $\{x_0, x_1, \ldots, x_{n-1}\}$ is an n-cycle for f, show that the derivative of f^n is the same at each x_i, $i = 0, 1, \ldots, n-1$.

9. Show that if x_0 is a peridic point of f with period n and $f^m(x_0) = x_0$ $m \in \mathbb{Z}^+$, then $m = kn$ for some $k \in \mathbb{Z}^+$. (Hint: Write $m = qn + r$ for some r, $0 \leq r < n$ and show that $r = 0$).

10. Show that if $f^p(x) = x$ and $f^q(x) = x$, and n is the highest common factor of p and q, then $f^n(x) = x$. (Hint: Use the previous question and the fact that every common factor of p and q is a factor of n).

11. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be an *odd function* ($f(-x) = -f(x)$ for all $x \in \mathbb{R}$). Show that $x = 0$ is a fixed point and that the intersection of the graph of f with the line $y = -x$ gives rise to points c of period 2 (when $c \neq 0$).

(b) Use part (a) to find the 2-cycles of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x/2$ and determine their stability.

(c)) Let $f : \mathbb{R} \to \mathbb{R}$ be an *even function* ($f(-x) = f(x)$ for all $x \in \mathbb{R}$). Show that the intersection of the graph of f with the line $y = -x$ gives rise to eventually fixed points c (when $c \neq 0$). What are the eventually fixed points of $f(x) = \cos(x)$?

12. (a) Let $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Find the fixed points of f and determine their nature/stability. Show that $x = 0$ is a *non-isolated* fixed point of f (there are fixed points of f that are arbitrarily close to 0). Find the eventual fixed points of f (note that $f(x)$ is an even function).

(b) (Open ended question) Discuss the stability of the fixed points of $f(x)$, their basins of attractions and the existence of period 2-points. Note that $f(x) \leq x$ for all $x \in \mathbb{R}$.

(c) Now set $g(x) = \begin{cases} x\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Find the fixed points of g, and show that in this case there are period 2-points (Hint: note that $g(x)$ is an odd function - see the previous question).

2.4 Periodic Points of the Logistic Map

We try to find the 2-cycles of the logistic map $L_\mu(x) = \mu x(1-x)$, $0 \leq x \leq 1$. To do this we solve the equation

$$L_\mu^2(x) = x,$$

or

$$\mu x(1-x)[1-\mu x(1-x)] - x = 0,$$

or

$$-\mu^3 x^4 + 2\mu^3 x^3 - (\mu^3+\mu^2)x^2 + \mu^2 x - x = 0,$$

Clearly x is a factor (since $c = 0$ is a fixed point of $L_\mu(x)$) and similarly $x - (1 - 1/\mu)$ must be a factor so we get

$$L_\mu^2(x) - x = -x(\mu x - \mu + 1)(\mu^2 x^2 - \mu(\mu+1)x + \mu + 1),$$

giving a quadratic equation which has no roots if $\mu < 3$:

$$\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0.$$

Solving using the quadratic formula gives

$$\begin{aligned} c &= \frac{\mu(\mu+1) \pm \sqrt{\mu^2(\mu+1)^2 - 4\mu^2(\mu+1)}}{2\mu^2} \\ &= \frac{(1+\mu) \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu}. \end{aligned}$$

This is real only for $\mu \geq 3$ (called the "birth of period two"). Let us call these two roots c_1 and c_2 (dependent on μ).

This 2-cycle is asymptotically stable if

$$|(L_\mu^2)'(c_1)| = |L_\mu'(c_1)L_\mu'(c_2)| < 1,$$

or

$$-1 < \mu^2(1-2c_1)(1-2c_2) < 1,$$
$$-1 < (-1 - \sqrt{(\mu^2 - 2\mu - 3)})(-1 + \sqrt{(\mu^2 - 2\mu - 3)}) < 1,$$

$$-1 < 1 - (\mu^2 - 2\mu - 3) < 1,$$

and this gives rise to the two inequalities

$$\mu^2 - 2\mu - 3 > 0 \quad \text{and} \quad \mu^2 - 2\mu - 5 < 0,$$

and solving gives

$$3 < \mu < 1 + \sqrt{6}.$$

This is the condition for asymptotic stability of the 2-cycle $\{c_1, c_2\}$.

For $\mu = 1 + \sqrt{6}$, it can be seen that

$$L'_\mu(c_1)L'_\mu(c_2) = -1, \quad \text{and} \quad SL^2_\mu(c_1) < 0,$$

so Theorem 1.5.7 (i) shows that the 2-cycle is asymptotically stable. Also, the 2-cycle is unstable for $\mu > 1 + \sqrt{6}$. In summary:

Theorem 2.4.1 *For $3 < \mu \leq 1 + \sqrt{6}$, the logistic map $L_\mu(x) = \mu x(1 - x)$ has an asymptotically stable 2-cycle. The 2-cycle is unstable for $\mu > 1 + \sqrt{6}$.*

The above shows that we have a bifurcation when $\mu = 3$ where a 2-cycle is created and which was not previously present. There is another bifurcation at $\mu = 1 + \sqrt{6}$. This means that for $3 < \mu \leq 1 + \sqrt{6}$, when we use graphical iteration of points close to c_1 and c_2, they will approach the period 2 orbit, and not the fixed point (which is now unstable). In fact it can be shown that for this range of values of μ, the basin of attraction of the 2-cycle consists of all of $(0, 1)$, (except for the fixed point $1 - 1/\mu$ and eventual fixed points such as $1/\mu$). When μ exceeds $1 + \sqrt{6} = 3.449499\ldots$, the period 2-points become unstable and this no longer happens, but we shall see that something different happens. We have another bifurcation when $\mu = 1 + \sqrt{6}$, with the birth of an attracting period 4-cycle.

2.5 The Period Doubling Route to Chaos

We summarize what we have determined so far and talk about what happens as μ increases from zero to around 3.57.

For $\mu < b_1 := 1$, $c = 0$ is the only fixed point and it is attracting for these values of μ. There is a bifurcation at $b_1 = 1$, where a non-zero fixed point $c = 1 - 1/\mu$ is created. This fixed point is attracting for $\mu \leq 3$ (and $c = 0$ is no longer attracting) and super attracting when $\mu = s_1 = 2$. The second bifurcation occurs when $\mu = b_2 = 3$. The fixed point $c = 1 - 1/\mu$ becomes unstable, and an attracting 2-cycle is created for $3 < \mu < 1 + \sqrt{6} = b_3$.

2.5.1 A Super-Attracting Period 2-Cycle

We saw that when $\mu = 2$, $c = 1/2$ is a super-attracting fixed point for L_μ. We now look for a super-attracting period 2-cycle for L_μ when $3 < \mu < 1 + \sqrt{6}$, as it illustrates an important general method that can be used for finding where period three is born:

Suppose that $\{x_1, x_2\}$ is a 2-cycle for the logistic map L_μ, which is super-attracting, then

$$x_1 = \mu x_2(1 - x_2), \quad \text{and} \quad x_2 = \mu x_1(1 - x_1),$$

so multiplying these equations together gives the equation

$$\mu^2(1 - x_1)(1 - x_2) = 1.$$

In addition we must have

$$(L_\mu^2)'(x_1) = L_\mu'(x_1)L_\mu'(x_2) = 0,$$

so that

$$\mu^2(1 - 2x_1)(1 - 2x_2) = 0.$$

Thus either $x_1 = 1/2$, or $x_2 = 1/2$, so suppose the former holds, then $x_2 = \mu/4$. Substituting into the first equation gives

$$\mu^2(1 - \mu/4)(1 - 1/2) = 1,$$

or

$$\mu^3 - 4\mu^2 + 8 = 0.$$

$\mu - 2$ must is a factor of this cubic, so we have

$$(\mu - 2)(\mu^2 - 2\mu - 4) = 0,$$

and this gives $\mu = 1 + \sqrt{5}$. We have shown:

Proposition 2.5.2 *When $\mu = s_2 = 1 + \sqrt{5}$, $\{1/2, \frac{1+\sqrt{5}}{4}\}$ is a super-attracting 2-cycle for L_μ.*

When μ exceeds $b_3 = 1 + \sqrt{6}$, the 2-cycle ceases to be attracting and becomes repelling. In addition, a 4-cycle is created which is attracting until μ exceeds a value b_4 when it becomes repelling and an attracting 8-cycle is created. This type of period doubling continues so that when μ exceeds b_n, an attracting 2^{n-1}-cycle is created until μ reaches b_{n+1}. These cycles become super attracting at some s_n ($b_n < s_n < b_{n+1}$). This behavior continues with 2^n-cycles for all $n \in \mathbb{Z}^+$ being created, until μ reaches

approximately 3.57. In other words, for $b_n < \mu < b_{n+1}$, L_μ has a stable 2^n-cycle. It can be shown that $b_\infty = \lim_{n\to\infty} b_n = 3.570$ approximately.

Although $b_{n+1} - b_n \to 0$ as $n \to \infty$, it can be shown that

$$\lim_{n\to\infty} \frac{b_n - b_{n-1}}{b_{n=1} - b_n} = \delta = 4.6692016\ldots,$$

(called Feigenbaums's number). Feigenbaum showed that you get the same constant δ in this way for any family of unimodal maps ($f_\mu(x)$ is *unimodal* if $f_\mu(0) = 0$, $f_\mu(1) = 0$, f_μ is continuous on $[0, 1]$ with a single critical point between 0 and 1).

2.6 The Bifurcation Diagram.

The behavior described above can be illustrated graphically using a *bifurcation diagram*. To create a bifurcation diagram we plot μ, $0 \leq \mu \leq 4$ along the x-axis, and values of $L_\mu^n(x)$ along the y-axis. The idea is to calculate (say) for each value of μ the first 500 iterates of some (arbitrarily chosen) point x_0. We ignore the first 450 iterates and plot the next 50. So for example, if $1 < \mu < 3$, because the fixed point is attracting, the iterates will approach the fixed point $1 - 1/\mu$, so for n large, what we see plotted will be (very close to) the value $1 - 1/\mu$. For $3 < \mu < 1 + \sqrt{6}$ the fixed point has become repelling, so this no longer shows up, but the 2-cycle has become attracting, so we see plotted the 2 points of the 2-cycle. This continues with the 4-cycle, 8-cycle etc. This is called the *period doubling route to chaos.*

We can create a bifurcation diagram for L_μ using Mathematica in the following way:

First we define the logistic map for $\mu = 1$, and then a function of two variables:

```
f[x_]:= x(1-x)

h[x_,a_]:= a*f[x]
```

For a given $a \in [0, 4]$ we pick $x \in (0, 1)$ randomly and iterate $h(x, a)$ 100 times. Since this gives rise to a single number, we can define a function as the output:

```
g[a_]:=(
  k[x_]:=h[x,a];
  A=NestList[k, Random[Real, {0,1}], 100];
```

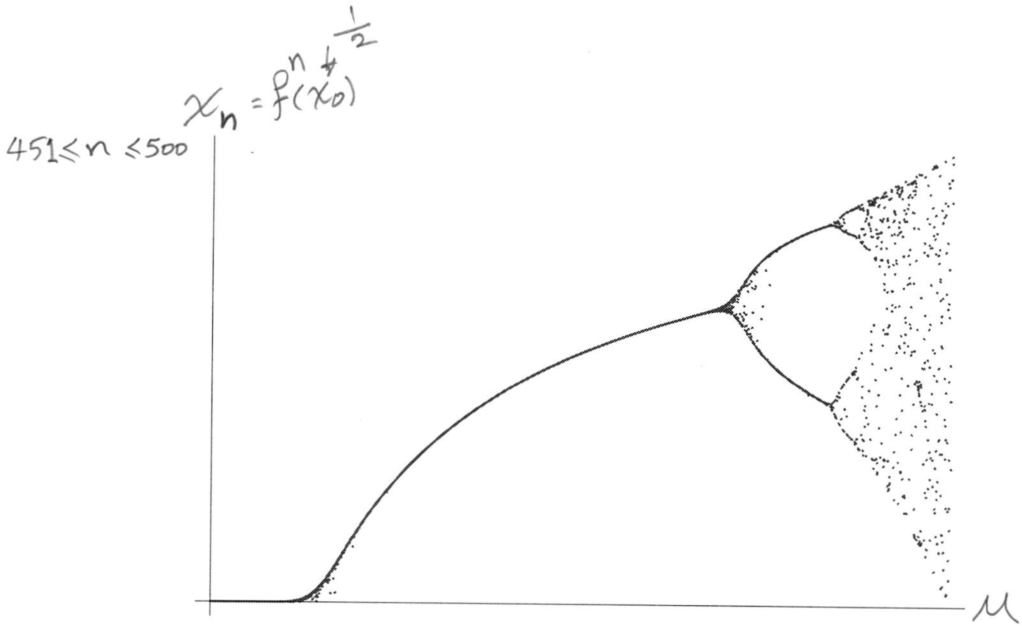
```
Return[A[[100]]]
)
```

Now we generate a list using the Table command for $a \in [1,4]$:

```
B=Table[g[j/1000],{j,1000,4000}];
```

and now we plot this list:

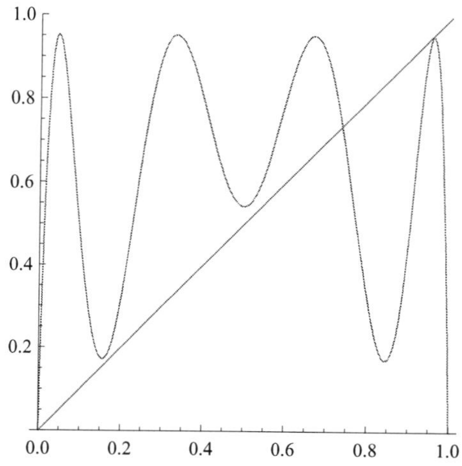
```
ListPlot[B, PlotStyle->PointSize[0.001],Ticks->False]
```



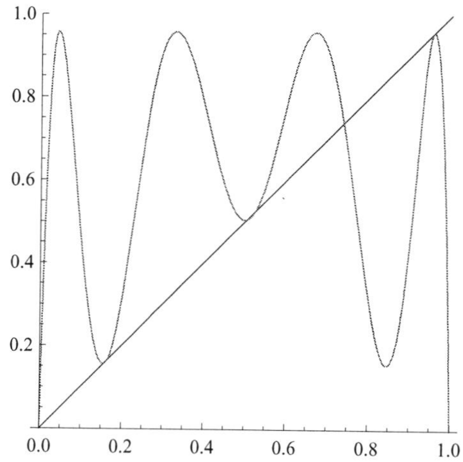
The bifurcation diagram for the logistic map.

2.6.1 Where Does Period Three Occur for the Logistic Map

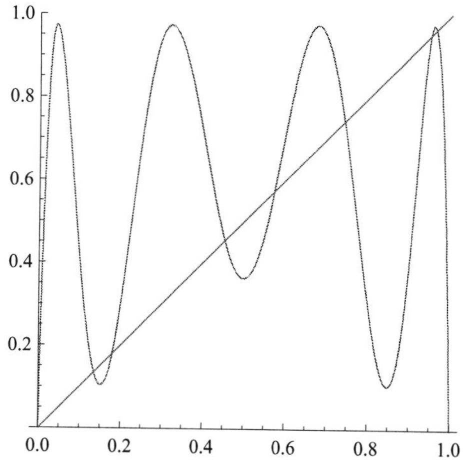
If we look at the graph of L_μ^3 for values of μ close to 3.8, we see that is where a 3-cycle is created:



The Graph of $L_\mu^3(x)$ for $\mu < 1 + \sqrt{8}$.



The Graph of $L_\mu^3(x)$ for $\mu = 1 + \sqrt{8}$.



The Graph of $L_\mu^3(x)$ for $\mu > 1+\sqrt{8}$.

We aim to show that this happens when $\mu = 1+\sqrt{8}$. For values of μ slightly smaller than this we only see the two fixed points. For $\mu = 1+\sqrt{8}$ we see that a 3-cycle is born, and for larger values of μ, we see two 3-cycles. For a small range of values of μ, the 3-cycle is attracting.

To show that the bifurcation occurs when $\mu = 1+\sqrt{8}$ we follow the argument of Feng [16] (see also [5], [17] and [42]):

Period 3-points of L_μ occur where $L_\mu^3(x) = x$, so we look at where

$$L_\mu^3(x) - x = 0.$$

To disregard the fixed points of L_μ we set

$$g_\mu(x) = \frac{L_\mu^3(x) - x}{L_\mu(x) - x}.$$

Using Mathematica (for example) we can check that

$$\begin{aligned} g_\mu(x) = \mu^6x^6 - (\mu^5+3\mu^6)x^5 + (\mu^4+4\mu^5+3\mu^6)x^4 - (\mu^3+3\mu^4+5\mu^5+\mu^6)x^3 \\ +(\mu^2+3\mu^3+3\mu^4+2\mu^5)x^2 - (\mu+2\mu^2+2\mu^3+\mu^4)x + 1 + \mu + \mu^2. \end{aligned}$$

Set $\lambda = 7 + 2\mu - \mu^2$ and let

$$h_\mu(z) = g_\mu(-z/\mu),$$

then

$$\begin{aligned} h_\mu(z) = z^6 + (3\mu+1)z^5 + (3\mu^2+4\mu+1)z^4 + (\mu^3+5\mu^2+3\mu+1)z^3 \\ +(2\mu^3+3\mu^2+3\mu+1)z^2 + (\mu^3+2\mu^2+2\mu+1)z + \mu^2 + \mu + 1. \end{aligned}$$

Then if

$$k_\mu(z) = \left\{ z^3 + z^2\frac{(3\mu+1)}{2} + z(2\mu + 3 - \frac{\lambda}{2}) + \frac{(\mu+5)}{2} - \frac{\lambda}{2} \right\}^2 + \frac{\lambda}{4}(z+1)^2(z+\mu)^2,$$

then we can check that $k_\mu(z) = h_\mu(z)$ for all z (e.g., using Mathematica).

Note that

$$\lambda > 0 \quad \text{for} \quad \mu < 1+\sqrt{8}, \quad \lambda = 0 \quad \text{for} \quad \mu = 1+\sqrt{8}, \quad \text{and} \quad \lambda < 0 \quad \text{for} \quad \mu > 1+\sqrt{8}.$$

This means that for $\mu < 1+\sqrt{8}$, $h_\mu(z) > 0$ for all z (we say $h_\mu(z)$ is *positive definite*), so cannot have any roots, i.e., $g_\mu(x) = 0$ has no solution, so L_μ cannot have any 3-cycles. We summarize this as follows:

Theorem 2.6.2 [16] (i) *If $0 < \mu < 1+\sqrt{8}$, then $h_\mu(z)$ is positive definite and the equation $h_\mu(z) = 0$ does not have any real roots. Consequently, the logistic map $L_\mu(x)$ does not have a 3-cycle.*

(ii) *If $\mu = 1+\sqrt{8}$, then $h_\mu(z) = 0$ has three distinct roots, each of multiplicity two. These three roots constitute a 3-cycle for $L_\mu(x)$.*

(iii) *If $\mu > 1+\sqrt{8}$ (with $\mu - (1+\sqrt{8})$ sufficiently small), the equation $h_\mu(z) = 0$ has six simple roots which give rise to two 3-cycles for $L_\mu(x)$.*

Proof. (i) If $\mu < 1+\sqrt{8}$, then since $h_\mu(z)$ is positive definite the result follows.

(ii) If $\mu = 1+\sqrt{8}$, then $\lambda = 0$ and the equation becomes

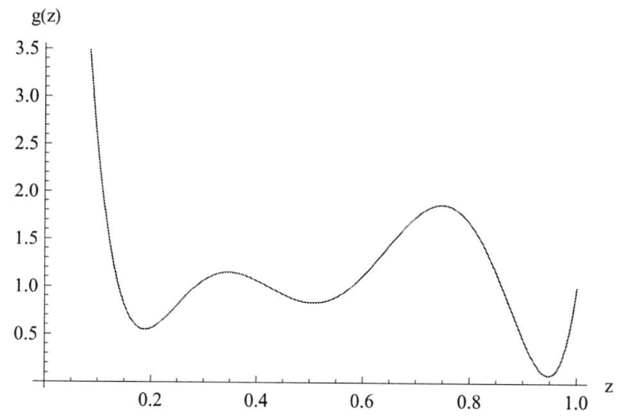
$$h_\mu(z) = \left(z^3 + (2+3\sqrt{2})z^2 + (5+4\sqrt{2})z + 3 + \sqrt{2}\right)^2.$$

The resulting cubic can be solved using the cubic formula to give three real solutions, z_1, z_2, z_3 and these can be used to give the three solutions to $g_\mu(x) = 0$, corresponding to the 3-cycle of $L_\mu(x)$:

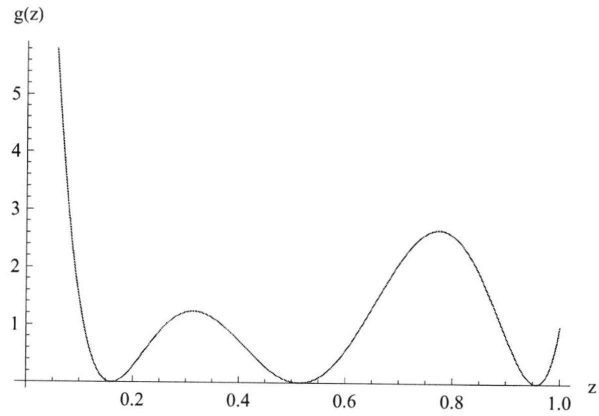
$$z_k = \frac{2\sqrt{7}}{3}\cos\left(\frac{1}{3}\arccos(-\frac{1}{2\sqrt{7}} + \frac{2k\pi}{3})\right) - \frac{2+3\sqrt{2}}{3}, \quad k = 0, 1, 2,$$

(see the graph of $g_\mu(x)$ below).

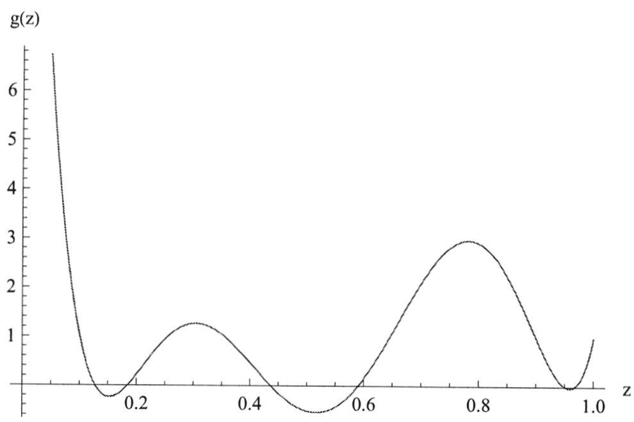
(iii) For $\lambda < 0$ we can factor $h_\mu(z) = h_1(z)h_2(z)$ using the difference of two squares, and then use the Intermediate Value Theorem on each of $h_1(z)$ and $h_2(z)$, to see that they each have three different roots corresponding to two 3-cycles, which can then be shown to be distinct (see [16] for details). $\square$



$g_{\mu}(x)$ for $\mu < 1+\sqrt{8}$.



$g_{\mu}(x)$ for $\mu = 1+\sqrt{8}$.



$g_{\mu}(x)$ for $\mu > 1+\sqrt{8}$.

Example 2.6.3: A Super-Attracting 3-Cycle for the Logistic Map

Recall that super-attracting periodic points occur where the derivative is zero, so for a super-attracting 3-cycle $\{c_1, c_2, c_3\}$, we require

$$(L_\mu^3)'(c_1) = L'_\mu(c_1)L'_\mu(c_2)L'_\mu(c_3) = 0,$$

i.e.,

$$(1 - 2c_1)(1 - 2c_2)(1 - 2c_3) = 0,$$

so we may assume $c_1 = 1/2$. This means that $x = 1/2$ is a solution of the equation $L_\mu^3(x) = x$, or $L_\mu^3(1/2) = 1/2$. But if μ satisfies the equation $L_\mu(1/2) = 1/2$, then it will also satisfy the equation involving the third iterate. Consequently we solve for μ: $g_\mu(1/2) = 1/2$, where $g_\mu(x) = (L_\mu^3(x) - x)/(L_\mu(x) - x)$ as defined in the last section (this eliminates the root $\mu = 2$ which gave rise to the superstable fixed point at $x = 1/2$). We obtain

$$\frac{1}{64}(64 + 32\mu + 16\mu^2 - 24\mu^3 - 4\mu^4 + 6\mu^5 - \mu^6) = 0.$$

Set

$$p(a) = a^6 - 6a^5 + 4a^4 + 24a^3 - 16a^2 - 32a - 64,$$

then Mathematica indicates that there is a single real root μ_0 larger than $1+\sqrt{8}$ with exact value

$$\mu_0 = \frac{1}{6}\left\{6 + 2\sqrt{3\left(11 + 2\cdot 2^{2/3}(25 - 3\sqrt{69})^{1/3} + 2\cdot 2^{2/3}(25 + 3\sqrt{69})^{1/3}\right)}\right\}.$$

Mathematica is able to solve this equation exactly because it can be reduced to a cubic (and then the cubic formula may be used):

Following Lee [26], replace a by $a + 1$ and check (using Mathematica) that

$$p(a + 1) = a^6 - 11a^4 + 35a^2 - 89 = b^3 - 11b^2 + 35b - 89,$$

a cubic in $b = a^2$ which can now be solved exactly for b, then for a, from which the original equation can be solved. It is seen that

$$\mu_0 = 3.8318740552\ldots.$$

The other period 3-points may now be found since $c_1 = 1/2$, $c_2 = L_{\mu_0}(1/2) = \mu_0/4 = 0.95796\ldots$, and $c_3 = L_{\mu_0}^2(1/2) = \mu_0^2/4(1 - \mu_0/4) = 0.15248\ldots$.

Example 2.6.4: The 3-Cycle when $\mu = 4$

When $\mu = 4$, $L_4(x) = 4x(1 - x)$. In this case we have

$$g_4(x) = 4096x^6 - 13312x^5 + 16640x^4 - 10048x^3 + 3024x^2 - 420x + 21,$$

and we can check using Mathematica (see also Lee [27])

$$\begin{aligned} g_4(x/4) &= x^6 - 13x^5 + 65x^4 - 157x^3 + 189x^2 - 105x + 21 \\ &= (x^3 - 7x^2 + 14x - 7)(x^3 - 6x^2 + 9x - 3), \end{aligned}$$

the product of two cubics. The solutions give a pair of 3-cycles, which we will show are given by the following theorem:

Theorem 2.6.5 *For the logistic map $L_4(x) = 4x(1-x)$ we have:*

(i) The 2-cycle is $\{\sin^2(\pi/5), \sin^2(2\pi/5)\}$

(ii) The 3-cycles are

$$\{\sin^2(\pi/7),\ \sin^2(2\pi/7),\ \sin^2(3\pi/7)\} \quad \textit{and} \quad \{\sin^2(\pi/9),\ \sin^2(2\pi/9),\ \sin^2(4\pi/9)\ \}.$$

Proof. Recall that the difference equation $x_{n+1} = 4x_n(1-x_n)$, $n = 0, 1, 2, \ldots$, has the solution

$$x_n = \sin^2\left(2^n \arcsin\sqrt{x_0}\right).$$

This was obtained by setting $x_n = \sin^2(\theta_n)$ for some $\theta_n \in (0, \pi/2]$, $(n = 1, 2\ldots)$, so that

$$\sin^2(\theta_{n+1}) = 4\sin^2(\theta_n)(1 - \sin^2(\theta_n)) = 4\sin^2(\theta_n)\cos^2(\theta_n) = \sin^2(2\theta_n).$$

We can use this to show that $\sin^2(\theta_{n+1}) = \sin^2(4\theta_{n-1})$, so in general

$$x_n = \sin^2(\theta_n) = \sin^2(2^n\theta_0), \quad \text{where} \quad \theta_0 = \arcsin(\sqrt{x_0}).$$

In particular we have

$$\theta_1 = \arcsin\left(\sqrt{\sin^2 2\theta_0}\right) = \begin{cases} 2\theta_0 & \text{for} \quad 0 \le \theta_0 \le \pi/4 \\ \pi - 2\theta_0 & \text{for} \quad \pi/4 \le \theta_0 \le \pi/2 \end{cases}$$

Applying this to the situation where we have a 2-cycle $\{c_0, c_1\}$, if $c_i = \sin^2(\theta_i)$, we get θ_0 is equal to $4\theta_0$, $\pi - 2\theta_0$, $2\pi - 4\theta_0$ or $\pi - 4\theta_0$. This gives $\theta = 0$ or $\theta = \pi/3$ (giving rise to the two fixed points) or $\theta_0 = \pi/5$, or $2\pi/5$ from which the result follows. A similar, but more complicated analysis gives the 3-cycles. $\square$

Remark 2.6.6 Using the above ideas, if we want to find the period n-points of L_4, we solve the equation $L_4^n(x) = x$. When $x = \sin^2(\theta)$, this becomes

$$\sin^2(\theta) = \sin^2(2^n\theta).$$

This gives rise to the two equations

$$\pm\theta = 2^n\theta + 2k\pi, \quad \text{or} \quad \pm\,\theta = (2k+1)\pi - 2^n\theta, \quad \text{for some} \quad k \in \mathbb{Z}.$$

This can be summarized as a single equation:

$$\pm\theta = 2^n\theta + k\pi \Rightarrow \theta = \frac{k\pi}{2^n \pm 1}, \quad n = 1, 2, 3 \ldots, \; k \in \mathbb{Z}$$

so that

$$\text{Per}_n(L_4) = \{\sin^2(\frac{k\pi}{2^n - 1}) : 0 \le k < 2^{n-1}\} \cup \{\sin^2(\frac{k\pi}{2^n + 1}) : 0 < k \le 2^{n-1}\}.$$

It follows that L_4 has points of all possible periods. We shall see that the set of all periodic points constitutes a "dense" set in $[0, 1]$ and each of these points is unstable.

Exercises 2.6

1. Recall the family of maps defined by $S_\mu(x) = \mu \sin(x)$ for $x \in [0, \pi]$ and $\mu \in [0, \pi]$. Use the Mathematica commands below together with the `ManipulatePlot` command to estimate the values of μ where periods two and three are born. Use

```
h[a_,x_]:= a*Sin[x]
g[a_,x_]:= h[a,h[a,x]]
k[a_,x_]:= h[a,h[a,h[a,x]]]


   Manipulate[Plot[{x,g[a,x]}, {x, 0, Pi}, PlotRange->{0,Pi},
AspectRatio->Automatic], {a,0,Pi}]
```

(Click on the + at the end of the `Manipulate` line to obtain the `Animation Controls`).

2. Modify the bifurcation diagram of the logistic maps to give a bifurcation diagram for the family S_μ, $0 \le \mu \le \pi$.

3. Do the same as in question 1 for the family $C_\mu(x) = \mu\cos(x)$, $x \in [-\pi, \pi]$ and $\mu \in [0, \pi]$.

4. Can you use the method of question 1 to estimate a value of μ for which S_μ has a superattracting 2-cycle, or 3-cycle?

5. Let $g_\mu(x) = \mu x \dfrac{(1-x)}{(1+x)}$, $\mu > 0$.

(a) Show that g_μ has a maximum at $x = \sqrt{2}-1$ and the maximum value is $\mu(3-2\sqrt{2})$.

(b) Deduce that g is a dynamical system on $[0,1]$ for $0 \le \mu \le 3+2\sqrt{2}$ (i.e., $g_\mu([0,1]) \subseteq [0,1]$).

(c) Find the fixed points of g_μ for $\mu \ge 1$.

(d) Give conditions on μ for the fixed points of g_μ to be attracting.

(e) Use Mathematica to graph g_μ^2 and g_μ^3, and estimate when a period 2-point is created.

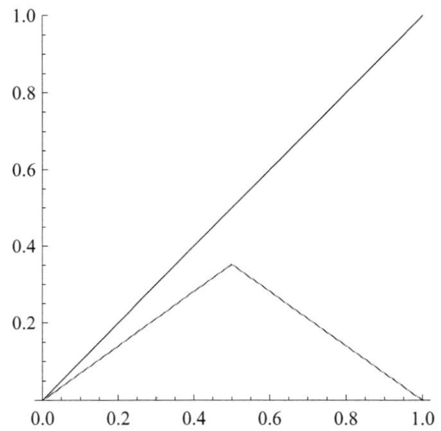
(f) Use Mathematica to give a bifurcation diagram for g_μ, for $0 \le \mu \le 3+2\sqrt{2}$.

2.7 The Tent Family T_μ.

We define another parameterized family in a piecewise linear manner: $T_\mu : [0,1] \to [0,1]$, $0 < \mu \le 2$,

$$T_\mu(x) = \begin{cases} \mu x & \text{if } \ 0 \le x \le 1/2 \\ \mu(1-x) & \text{if } \ 1/2 < x \le 1. \end{cases}$$

When $\mu = 2$ we get the familiar tent map $T(x)$ seen earlier. We now look at the parameter values $0 < \mu \le 2$ (and later we shall see what happens when $\mu > 2$, and where $T_\mu : \mathbb{R} \to \mathbb{R}$). If $0 < \mu < 1$ we see that the only fixed point is $c = 0$. If $\mu = 1$, then all $c \in [0, 1/2]$ are fixed points and if $1 < \mu \le 2$; then there are 2 fixed points. We look at each of these cases separately.



Case 2.7.1 $0 < \mu < 1$. We see that 0 is the only fixed point of T_μ and if $0 < x \le 1/2$, then $0 \le T_\mu(x) = \mu x < x$, and if $1/2 < x \le 1$, then

$$0 \le T_\mu(x) = \mu(1-x) < 1 - x < \frac{1}{2} < x,$$

so in a similar manner to what we have seen previously, the sequence $T_\mu^n(x)$ is decreasing, bounded below by 0, so must converge to the fixed point 0. Thus $B_{T_\mu}(0) = [0,1]$.

Case 2.7.2 $\mu = 1$ Clearly if $0 < x \le 1/2$, then $T_1(x) = x$, so is a fixed point, and if $1/2 < x \le 1$, then x is an eventual fixed point since

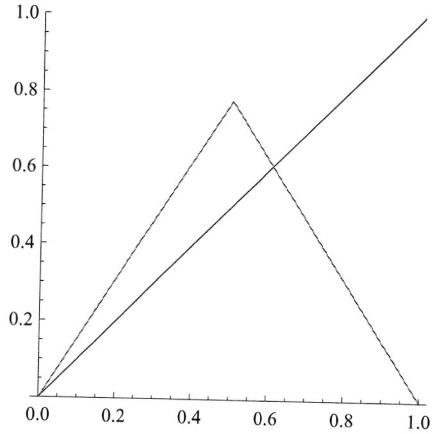
$$T_1^2(x) = T_1(1-x) = x.$$

The fixed points are stable but not attracting.

Case 2.7.3 $1 < \mu < 2$. There is a bifurcation at $\mu = 1$ with a second fixed point c created, $1/2 < c < 1$. We solve $T_\mu(x) = x$ to find this point:

$$T_\mu(x) = \mu(1-x) = x, \quad \text{so that} \quad c = \frac{\mu}{1+\mu}.$$

Since $|T_\mu(x)| = \mu > 1$ in this case, the fixed point is repelling.



Case 2.7.4 $\mu = 2$. In this case $T_2 = T$, the familiar tent map. The fixed points 0 and 2/3 are repelling. This time the range of T is all of $[0,1]$. T has the effect of mapping the interval $[0, 1/2]$ onto all of $[0,1]$ and then folding the interval $[1/2, 1]$ back over the interval $[0,1]$. It is this stretching and folding that gives rise to the chaotic nature of T that we will see later, and is typical of many transformations of this type. We will examine the periodic points of T in the next section.

2.8 The 2-Cycles and 3-Cycles of the Tent Family

At some stage for $\mu \geq 1$ a 2-cycle is created. It is interesting to use Mathematica to do a dynamic iteration of T_μ to see how this happens (use the `ManipulatePlot` command described in Exercises 2.6, but with

```
h[x_,a_]:= a*Piecewise[{{x, x < 1/2}, {(1-x), 1/2 < x}}]
```

in place of `a*Sin[x]`). It can be checked that for $\mu > 1$, T_μ^2 is given by the formula

$$T_\mu^2(x) = \left\{ = \begin{array}{lll} \mu^2 x & \text{if} & 0 \leq x \leq \frac{1}{2\mu} \\ \mu(1-\mu x) & \text{if} & \frac{1}{2\mu} < x \leq \frac{1}{2} \\ \mu(1-\mu+\mu x) & \text{if} & \frac{1}{2} < x \leq 1 - \frac{1}{2\mu} \\ \mu^2(1-x) & \text{if} & 1 - \frac{1}{2\mu} < x \leq 1 \end{array} \right.$$

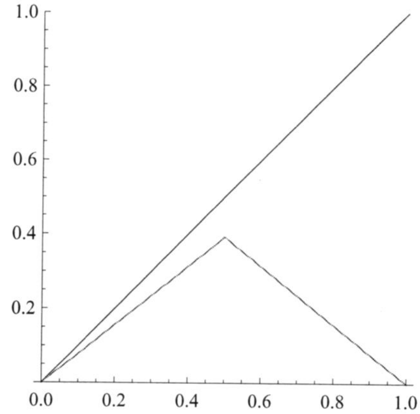
The 2-cycle is created when $T_\mu^2(1/2) = 1/2$, i.e., when

$$\mu(1-\mu/2) = 1/2, \quad \text{or} \quad (\mu-1)^2 = 0, \quad \mu = 1,$$

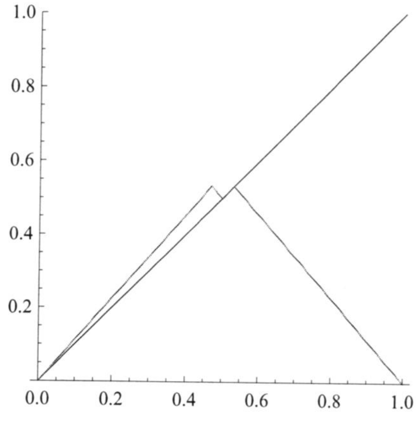
so the 2-cycle appears when $\mu > 1$. For $\mu \leq 1$, there are no period 2 points. The 2-cycle $\{c_1, c_2\}$ say, is unstable since

$$|(T_\mu^2)'(c_1)| = |T_\mu'(c_1)T_\mu'(c_2)| = \mu^2 > 1.$$

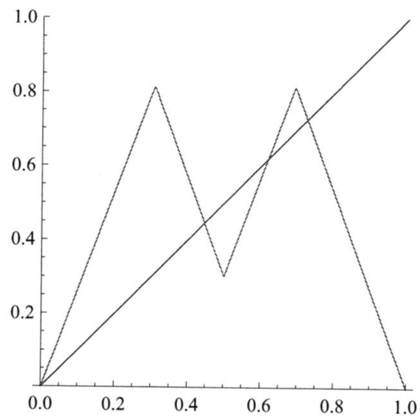
For example, if we solve $\mu(1-\mu x) = x$, we get $c_1 = \frac{\mu}{1+\mu^2}$ and solving $\mu^2(1-x) = x$ gives $c_2 = \frac{\mu^2}{1+\mu^2}$ as the period 2-points. Solving $\mu(1-\mu+\mu x) = x$ gives $c = \frac{\mu}{1+\mu}$ as the (non-zero) fixed point.



T_μ^2 for $\mu < 1$.



T_μ^2 for $\mu > 1$ (but close to 1).



T_μ^2 for $\mu > 1$.

We now look for the smallest value of μ for which there exists a 3-cycle. In a similar way to the situation for the 2-cycle, we look for where $T_\mu^3(1/2) = 1/2$ and we compute the value of the largest root of this equation. In general (see Heidel [19]) the smallest value of μ for which there exists a periodic orbit of period k is precisely the value of μ for which $1/2$ has period k. Using the formula for $T_\mu^2(1/2) = \mu(1 - \mu/2)$ above, then since $\mu(1-\mu/2) \leq 1/2$ for all μ, we get

$$T_\mu^3(1/2) = \mu^2(1-\mu/2) = 1/2 \quad \text{when} \quad \mu^3 - 2\mu^2 + 1 = 0,$$

or $(\mu - 1)(\mu^2 - \mu - 1) = 0$. Disregarding $\mu = 1$ and solving the quadratic gives $\mu = (1+\sqrt{5})/2$ as the value of μ where period 3 first occurs.

In general it can be shown that for $k > 3$ odd, period k first occurs when μ is equal to the largest real root of the equation

$$\begin{aligned} &\mu^k - 2\mu^{k-1} + 2\mu^{k-3} - 2\mu^{k-4} + \ldots - 2\mu + 1 \\ &= (\mu - 1)(\mu^{k-1} - \mu^{k-2} - \mu^{k-3} + \mu^{k-4} - \mu^{k-5} + \mu^{k-6} \cdots + \mu - 1) = 0, \end{aligned}$$

so has $\mu - 1$ as a factor (see [19]). We will return to look at the tent family in Section 6.4.

Exercises 2.8

1. Find $\mu \in [0, 2]$ such that $c = 1/2$ is periodic of period 3 under the tent map T_μ (so that $\{1/2, \mu/2, \mu(1-\mu/2)\}$ is a 3-cycle).

2. For $\mu > 1$, show that T_μ has no attracting periodic points.

3. If $L_\mu(x) = \mu x(1-x)$ is such that $c = 1/2$ is n-periodic for some $n \in \mathbb{Z}^+$, prove that $1/2$ is an attracting periodic point. Is it necessarily a super-attracting periodic point?

4. Modify the bifurcation diagram of the logistic maps to give a bifurcation diagram for the tent family T_μ, $x \in [0,1]$ and $\mu \in [0,2]$.

3.1 Period 3 Implies Chaos. In 1975 in a paper entitled "Period three implies chaos", Li and Yorke proved a remarkable theorem:

Theorem 3.1.1 *Let $f : X \to X$ be a continuous function defined on an interval $X \subseteq \mathbb{R}$. If $f(x)$ has a point of period three, then for any $k = 1, 2, 3, \ldots$, there is a point having period k.*

This paper stirred considerable interest and shortly after it was pointed out that a Ukrainian mathematician by the name of Sharkovsky had in 1964 published a much more general theorem (in Russian) in a Ukrainian journal. His theorem was unknown in the west until the appearance of the Li-Yorke Theorem. To state his theorem we need to define a new ordering of the positive integers $\mathbb{Z}^+$. In the "Sharkovsky ordering", 3 is the largest number, followed by 5 then 7 (all of the odd integers), then $2 \cdot 3$, $2 \cdot 5$, (2 times the odd integers), then 2^2 times the odd integers etc., finishing off with powers of 2 in descending order:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd \cdots \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd \cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Sharkovsky's Theorem says that if a continuous map has a point of period k, then it has points of all periods less than k in the Sharkovsky ordering. The converse is also true in the sense that for each $k \in \mathbb{Z}^+$ there is a map having points of period k, but no points of period larger than k in the Sharkovsky ordering.

Theorem 3.1.2 (Sharkovsky's Theorem, 1964.) *Let $f : X \to X$ be a continuous map on an interval X (where X may be any bounded or unbounded subinterval of $\mathbb{R}$). If f has a point of period k, then it has points of period r for all $r \in \mathbb{Z}^+$ with $k \rhd r$.*

For example, this theorem tells us that if f has a 4-cycle, then it also has a 2-cycle and a 1-cycle (fixed point). If f has a 3-cycle, then f has all other possible cycles. If f has a 6-cycle, then since $6 = 2 \cdot 3$, f will have $2\cdot 5, 2\cdot 7, \ldots 2^2\cdot 3, 2^2\cdot 5 \ldots$-cycles, etc.

We omit the proof of this theorem, but shall prove the case where $k = 3$ (often known as the Li-Yorke Theorem because of their independent proof of this result in 1975). James Yorke is a professor at UMD, College Park and Li was his graduate student. A few years ago, Yorke and Benoit Mandelbrot were awarded the Japan Prize (the Japanese equivalent of the Nobel Prize) for their work in Dynamical Systems and Fractals (Mandelbrot is regarded as the "father" of fractals, and we will review some

of his work later). In order to prove this result we shall need some preliminary lemmas. The first was proved in Chapter 1 (Theorem 1.2.9).

Lemma 3.1.3 *Let $f : I \to \mathbb{R}$ be a continuous map, I an interval with $J = f(I) \supseteq I$, then $f(x)$ has a fixed point in I.*

Lemma 3.1.4 *Let $f : I \to \mathbb{R}$ be a continuous map. If $J \subseteq f(I)$ is a closed bounded interval, then there exists a closed bounded interval $K \subseteq I$ with $f(K) = J$.*

Proof. Write $J = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$.

There are $r, s \in I$ with $f(r) = a$ and $f(s) = b$. Set

$$\beta = y, \quad \text{where} \quad y \in I, \quad |y - r| \quad \text{is a minimum, and} \quad f(y) = b,$$

(β exists from continuity and $f(\beta) = b$: β is the point of I that is closest to r with the property that $f(\beta) = b$). Similarly, set

$$\alpha = x, \quad \text{where } x \text{ lies between } \beta \text{ and } r, \quad |x - \beta| \quad \text{is a minimum, and} \quad f(x) = a$$

(possibly equal to r), so as before $f(\alpha) = a$. In this way, there is no $c \in (\alpha, \beta)$ where $f(c) = a$, or $f(c) = b$. Then the closed interval K (either $K = [\alpha, \beta]$ or $K = [\beta, \alpha]$), has the property $f(K) = [a, b]$, since firstly, $f(\alpha) = a$, $f(\beta) = b$, so by the Intermediate Value Theorem $[a, b] \subseteq f(K)$. On the other hand, if $w \in f(K)$, then $w = f(z)$ for some z between α and β. This must give $f(z) \in [a, b]$ since otherwise, another application of the Intermediate Value Theorem would contradict the choices of α and β. $\square$

3.1.5 Proof of Sharkovsky's Theorem for $k = 3$

We are assuming that f has a point of period 3, so there is a 3-cycle $\{a, b, c\}$ with $f(a) = b$, $f(b) = c$ and $f(c) = a$. We assume that $a < b < c$ (the other case with $a < c < b$ may be treated similarly).

We give the idea of the proof by showing why there must be points of period one, two and four. Let

$$[a, b] = L_0 \quad \text{and} \quad [b, c] = L_1.$$

Observe that

$$f(L_0) \supseteq L_1 \quad \text{and} \quad f(L_1) \supseteq L_0 \cup L_1.$$

Case 1. f **has a fixed point.**

Since

$$f(L_1) \supseteq L_0 \cup L_1 \supseteq L_1,$$

Lemma 3.1.3 implies that f has a fixed point in L_1.

Case 2. f has a point of period 2.

This time we use

$$f(L_1) \supseteq L_0 \cup L_1 \supseteq L_0,$$

so by Lemma 3.1.4 there is a set $B \subseteq L_1$ such that $f(B) = L_0$. We then have

$$f^2(B) = f(L_0) \supseteq L_1 \supseteq B,$$

so by Lemma 3.1.3, B contains a fixed point c of f^2. c is a period 2-point of f (and not a fixed point of f) because

$$f(c) \in L_0 \quad \text{and} \quad c \in L_1, \quad \text{so} \quad f(c) \neq c.$$

Case 3. f has a point of period 4.

The above two constructions do not illustrate the general method, but the following construction is easily generalized to any number of fixed points greater than 3. Our aim is to show that there is a subset B of L_1 which is mapped first by f into L_1, then into L_1 again, then onto L_0 and then onto L_1, so that $f^4(B) \supseteq B$. Thus f^4 has a fixed point c in B, which cannot be a point of lesser period because $f(c) \in L_1$, $f^2(c) \in L_1$, $f^3(c) \in L_0$ and $f^4(c) \in B$ (so cannot have $f(c) = c$, $f^2(c) = c$ or $f^3(c) = c$).

It is useful to think of 5 copies of $L_0 \cup L_1$ with f mapping the first to the second etc. as shown:

We find sets B_1, B_2 and B_3 as follows:

$$f(L_0) \supseteq L_1, \quad f(L_1) \supseteq L_0 \cup L_1,$$

so there exists $B_1 \subseteq L_1$ such that $f(B_1) = L_0$.

There exists $B_2 \subseteq L_1$ such that $f(B_2) = B_1$ and there exists $B_3 \subseteq L_1$ such that $f(B_3) = B_2$. Set $B = B_3$, then

$$f^2(B_3) = f(B_2) = B_1, \quad \text{and so} \quad f^3(B) = L_0, \;\; f^4(B) \supseteq L_1 \supseteq B_3.$$

In other words $f^4(B) \supseteq B$, so there exists $c \in B$, a fixed point of f^4, which is not a point of period 3 or less, so must be a point of period 4. $\square$

In general if a function has points of period 4 the most we can deduce is that there are points of period 2 and fixed points. However, the following is true:

Proposition 3.1.6 *If $f : I \to I$ is continuous on an interval I with*

$$f(a) = b, \;\; f(b) = c, \;\; f(c) = d, \;\; f(d) = a, \quad a < b < c < d,$$

then $f(x)$ has a point of period 3, so also has points of all other periods.

Proof. We may assume that

$$f[a, b] = [b, c], \;\; f[b, c] = [c, d], \;\; f[c, d] = [a, d].$$

In particular, there exists $B_1 \subseteq [c, d]$ with $f(B_1) = [c, d]$. There exists $B_2 \subseteq [c, d]$ with $f(B_2) = [b, c]$.

Take $K_1 \subseteq B_1$ with $f(K_1) = B_2$, then

$$f^3(K_1) = f^2(B_2) = f[b, c] = [c, d] \supseteq K_1,$$

so f^3 has a fixed point in K_1 which is not a fixed point of $f(x)$.

$\square$

The above proofs can be summarized with the following type of result:

Proposition 3.1.7 *Let $f : I \to I$ be a continuous map, and let I_1 and I_2 be two closed subintervals of I with at most one point in common. If $f(I_1) \supset I_2$ and $f(I_2) \supset I_1 \cup I_2$, then f has a 3-cycle.*

Proof. Exercise: Use the ideas of the proof of 3.1.5. $\square$

3.2 Converse of Sharkovsky's Theorem

As we mentioned, for each $m \in \mathbb{Z}^+$ in the Sharkovsky ordering of $\mathbb{Z}^+$, Sharkovsky showed that there is a continuous map $f : I \to I$ (I an interval), such that $f(x)$ has a point of period m, but no point of period k for $k \rhd m$. The following were shown (as usual, I is either the real line or an interval):

Theorem 3.2.1 *For every $k \in \mathbb{Z}^+$, there exists a continuous map $f : I \to I$ that has a k-cycle, but has no cycles of period n for any n appearing before k in the Sharkovsky ordering.*

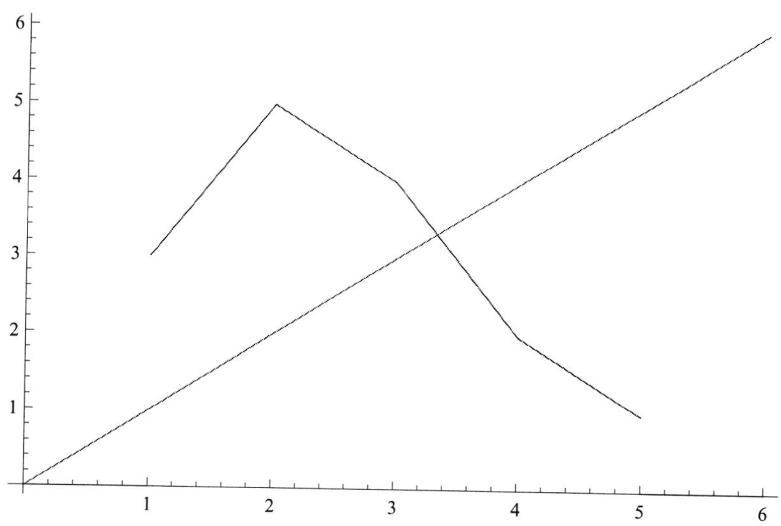
Theorem 3.2.2 *There exists a continuous map $f : I \to I$ that has a 2^n-cycle, for every $n \in \mathbb{Z}^+$ and has no other cycles of any other period.*

Strictly speaking, Sharkovsky's Theorem is really the combination of Theorem 3.1.2, Theorem 3.2.1 and Theorem 3.2.2 (see [32]), and sometimes the latter two theorems are referred to as the converse of Sharkovsky's Theorem (see [15]). We look at some particular cases of this:

Example 3.2.3 Define a function $f : [1, 5] \to [1, 5]$ as shown by the graph below (so

$$f(1) = 3, \quad f(2) = 5, \quad f(3) = 4, \quad f(4) = 2, \quad f(5) = 1,$$

with $f(x)$ piecewise linear between these points). Then f has a point of period 5, but no point of period 3.



Proof. Clearly no point from the set $\{1, 2, 3, 4, 5\}$ is of period 3, but this set is a 5-cycle. Also, Theorem 1.2.7 tells us that $f(x)$ has a fixed point c, so c is a fixed point for f^3. We shall show that f^3 has no other fixed points. Suppose to the contrary that

f^3 has another fixed point α. Now we can check that:

$$f^3[1,2] = [2,5], \quad f^3[2,3] = [3,5], \quad f^3[4,5] = [1,4],$$

so f^3 cannot have a fixed point in the intervals $[1,2]$, $[2,3]$ or $[4,5]$, so α must lie in the interval $[3,4]$. In fact $f^3[3,4] = [1,5] \supseteq [3,4]$ and we show that f^3 cannot have another fixed point in $[3,4]$.

If $\alpha \in [3,4]$, then $f(\alpha) \in [2,4]$, so either $f(\alpha) \in [2,3]$ or $f(\alpha) \in [3,4]$. If the former holds, then $f^2(\alpha) \in [4,5]$ and $f^3(\alpha) \in [1,2]$ which is impossible as we have to have $f^3(\alpha) = \alpha \in [3,4]$.

Thus we must have $f(\alpha) \in [3,4]$, so that $f^2(\alpha) \in [2,4]$. Again there are two possibilities: if $f^2(\alpha) \in [2,3]$ then $f^3(\alpha) \in [4,5]$, another contradiction, so that $f^2(\alpha) \in [3,4]$.

We have shown that the orbit of α: $\{\alpha, f(\alpha), f^2(\alpha)\}$ is contained in the interval $[3,4]$. On the interval $[3,4]$ we can check that $f(x)$ is given by the straight line formula

$$f(x) = 10 - 2x, \quad \text{and} \quad f(10/3) = 10/3,$$

so $c = 10/3$ is the unique fixed point of f. Also

$$f^2(x) = -10 + 4x, \quad f^3(x) = 30 - 8x,$$

also with the unique fixed point $x = 10/3$. It follows that f cannot have any points of period 3. $\square$

Remark 3.2.2 The above can be directly generalized to give a map having points of period $2n+1$, but no points of period $2n-1$, $n = 2, 3, \ldots$.

Exercises 3.2

1. Use the ideas of Chapter 3 to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a 2-cycle $\{a, b\}$, then f has fixed point.

2. Show that the map $f(x) = (x - 1/x)/2$, $x \neq 0$, has no fixed points but it has period 2-points. Find the 2-cycle and by looking at the graph of $f^3(x)$, check to see whether it has a 3-cycle. Why does this not contradict Sharkovsky's Theorem?

3. A map $f : [1, 7] \to [1, 7]$ is defined so that $f(1) = 4, f(2) = 7, f(3) = 6, f(4) = 5, f(5) = 3, f(6) = 2, f(7) = 1$, and the corresponding points are joined so the map is piecewise linear. Show that f has a 7-cycle but no 5-cycle.

4. If $F_\lambda(x) = 1 - \lambda x^2$ for $x \in \mathbb{R}$, show

(i) F_λ has fixed points for $\lambda \geq -1/4$.

(ii)F_λ has a 2-cycle for $\lambda > 3/4$.

(iii) The 2-cycle is attracting for $3/4 < \lambda < 5/4$.

5. Show that an increasing function $f : \mathbb{R} \to \mathbb{R}$ cannot have a 3-cycle. Can it have a 2-cycle? Answer the same questions when f is decreasing.

ch4 Metric Spaces من ص 73 لغاية ص 104 للاطلاع (سبق دراسته)

من هنا الشابتر الثالث

Chapter 5. Devaney's Definition of Chaos

In this chapter we will give Devaney's definition of chaos for one-dimensional maps and also for more general maps defined on metric spaces. One-dimensional maps are functions $f : I \to \mathbb{R}$ for some interval $I \subseteq \mathbb{R}$. It turns out that maps whose periodic points form a dense set often have highly chaotic properties.

5.1 The Doubling Map and the Angle Doubling Map

Example 5.1.1 The Doubling Map.

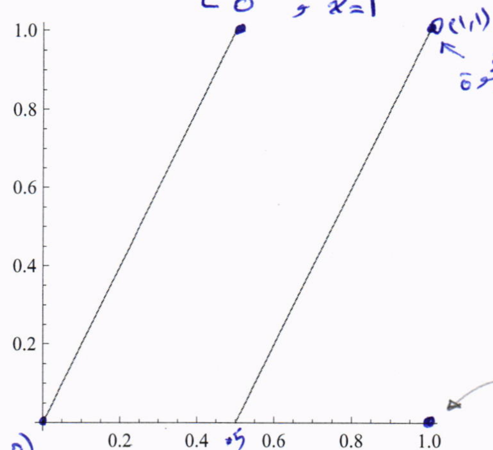
The doubling map $D : [0, 1] \to [0, 1]$ is defined by

$$D(x) = 2x(\text{mod } 1) = \begin{cases} 2x\ ; & 0 \le x < 1/2, \\ 2x - 1\ ; & 1/2 \le x \le 1. \end{cases}$$

0 ; $x=1$

أما بدون استخدام (mod) تسمى Baker map

$$B(x) = \begin{cases} 2x; & [0,\frac{1}{2}) \\ 2x-1; & [\frac{1}{2},1] \end{cases}$$



The doubling map D.

It is useful to describe D in terms of the binary expansion of a real number in [0,1]. Let $x \in [0, 1]$ with binary expansion

$$x = \cdot a_1 a_2 a_3 \dots \quad \text{where} \quad a_i = 0 \ \text{or} \ 1.$$

In other words,

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

Suppose that $a_1 = 0$ in the binary expansion of x, then

$$D(x) = 2x = \frac{a_2}{2} + \frac{a_3}{2^2} + \dots = \cdot a_2 a_3 \dots .$$

On the other hand, if $a_1 = 1$, then

$$D(x) = 2x - 1 = (a_1 + \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots) - 1 = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots = \cdot a_2 a_3 \ldots .$$

We see that in general

$$D(\cdot a_1 a_2 a_3 \ldots) = \cdot a_2 a_3 \ldots \quad \text{and} \quad D^n(\cdot a_1 a_2 a_3 \ldots) = \cdot a_{n+1} a_{n+2} a_{n+3} \ldots .$$

Consequently, if $x = \cdot a_1 a_2 \ldots a_n a_1 a_2 \ldots a_n a_1 \ldots$ has an expansion which repeats every n places, then $D^n(x) = x$, so that x is periodic of period n.

For example $D^2(\cdot 010101\ldots) = \cdot 010101\ldots$, so is a point of period 2. We use this to show that the set of periodic points of D are dense in $[0, 1]$. This can also be used to count the number of periodic points of period n. Notice that since $D'(x) > 1$ everywhere it is defined, all of the periodic orbits of D are unstable.

Proposition 5.1.2 *The periodic points of the doubling map are dense in* $[0, 1]$.

Proof. Let $\epsilon > 0$ and choose N so large that $1/2^N < \epsilon$. If $x \in [0, 1]$ it suffices to show that there is a periodic point y for D that is within ϵ of x. Suppose that the binary expansion of x is

$$x = \cdot a_1 a_2 a_3 \ldots = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

1) $\sum_{n=0}^{m} z^n = \frac{1 - z^{m+1}}{1 - z}; \; z \neq 1$

2) $\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}; \; |z| < 1$

then we set

$$y = \cdot a_1 a_2 \ldots a_N a_1 a_2 \ldots a_N a_1 \ldots ,$$

a point of period N. Then

So $\sum_{j=N+1}^{\infty} \frac{1}{2^j} = \sum_{j=0}^{\infty} \frac{1}{2^j} - \sum_{j=0}^{N} \frac{1}{2^j}$

$$|x - y| = |\sum_{j=N+1}^{\infty} \frac{b_j}{2^j}| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^N} < \epsilon,$$

$= \frac{1}{1 - \frac{1}{2}} - \frac{1 - (\frac{1}{2})^{N+1}}{1 - \frac{1}{2}}$

$= 2\left(1 - 1 + \frac{1}{2^{N+1}}\right) = \frac{1}{2^N}$

(where $b_i = 0, 1$ or -1).

Example 5.1.2 The Angle Doubling Map.

$f(z) = z^2 \quad f : S^1 \longrightarrow S^1 = \{z \mid |z| = 1\}$

As before, we denote by $\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$, the set of all complex numbers. If $z = a + ib \in \mathbb{C}$, then its absolute value (or modulus) is given by $|z| = \sqrt{a^2 + b^2}$. The conjugate of z is $\bar{z} = a - ib$ and we can check that $z\bar{z} = |z|^2$. We can represent $\mathbb{C}$ using points in the *(complex) plane* $\{(a, b) : a, b \in \mathbb{R}\}$. The *unit circle* S^1 in the complex plane is the set

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Points in S^1 may be represented as:

$$z = e^{i\theta} = \cos\theta + i \sin\theta, \quad \text{for some} \quad \theta \in \mathbb{R}.$$

(أو نستنتج) بطرحهما ما يلي:

3) $\sum_{k=n+1}^{\infty} z^k = \frac{z^{n+1}}{1 - z}$

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$n = -1, 0, 1, \ldots$

Here θ is the *argument* of z (written $\mathrm{Arg}(z)$), and it is the angle subtended by the ray from $(0,0)$ to (a,b) (when $z = a + ib$, $a, b \in \mathbb{R}$), and the *real axis.*

S^1 is a metric space if the distance between two points $z, w \in S^1$ is defined to be the shortest distance between the two points, going around the circle. We define a map $f : S^1 \to S^1$ by $\boxed{f(z) = z^2.}$ This map is called the *angle doubling map* because of the effect it has on $\theta = \mathrm{Arg}(z)$: $f(e^{i\theta}) = e^{2i\theta}$. We see that the angle θ is doubled. It is clear that there are a lot of similarities between the doubling map and the angle doubling map, and we shall examine this in the next section when we study the notion of conjugacy for dynamical systems. First we show that the periodic points of f are dense in S^1.

Consider the periodic points of $f(z) = z^2$: Solving $z^2 = z$ gives $z = 1$ (we can disregard $z = 0$), $f^2(z) = z$ gives $z^4 = z$ or $z^3 = 1$ and continuing in this way we see that the periodic points are certain nth roots of unity.

Proposition 5.1.3 *The periodic points of the angle doubling map $f : S^1 \to S^1$ are dense in S^1.*

$z^{2^n} \cdot z^{-1} = z\bar{z}^{-1}$

(الجذر النوني للـ 1)
القانون العام

Proof. If $f^n(z) = z$ for some $n \in \mathbb{Z}^+$, then $z^{2^n} = z$ or $z^{2^n - 1} = 1$. Write $z = e^{i\theta}$, then we want to find the $(2^n - 1)$th roots of unity. This gives:

$Z = \sqrt[n]{1}$ من الجذور n

$Z = \sqrt[n]{+1} \equiv Z^n = 1$ in $\mathbb{C}$

Sol. $Z_k = e^{\frac{2\pi k}{n} i}$ $K = 0, 1, \dots, n-1$ n-roots

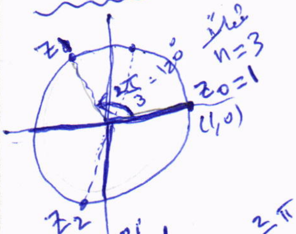
$\therefore Z_k = e^{\frac{2k\pi}{n} i} = e^{\frac{360^\circ k}{n} i}$

$$e^{(2^n-1)i\theta} = e^{2k\pi i}, \quad \text{for some} \quad k \in \mathbb{Z}^+,$$

$K = 0, 1, \dots, n-1$

giving the $\boxed{2^n - 1}$ distinct roots: $z_k = e^{2k\pi i/(2^n-1)}$, $k = 0, 1, 2, \dots, 2^n - 2$, showing that

$$\mathrm{Per}_n(f) = \{e^{2k\pi i/(2^n-1)} : 0 \le k < 2^n - 1\}, \quad n \in \mathbb{Z}^+.$$



These points are equally spaced around the circle, a distance $2\pi/(2^n - 1)$ apart, which can be made arbitrarily small by taking n large enough. It follows that the periodic points are dense in S^1. $\square$

$Z_0 = e^{0i} = 1$

$Z_1 = e^{\frac{2}{3}\pi i} = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$

$Z_2 = e^{\frac{4}{3}\pi i}$ 240°

5.2 Transitivity

Sometimes, given $f : X \to X$ (X a metric space), when we iterate $x_0 \in X$, the orbit $O(x_0) = \{x_0, f(x_0), \dots\}$, spreads itself evenly over X, so that $O(x_0)$ is a dense set in X. This leads to:

بشكل عام حل

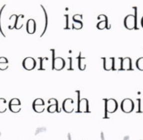
$Z^m = re^{i\theta}$

Definition 5.2.1 $f : X \to X$ is said to be *(topologically) transitive* if there exists $x_0 \in X$ such that $O(x_0)$ is a dense subset of X. A *transitive point* for f is a point x_0 which has a dense orbit under f. If f is transitive, then there is a dense set of transitive points, since each point in $O(x_0)$ will be a transitive point.

هذا التعريف موجود ايضا ص 85

$\therefore Z_k = \sqrt[m]{r}\, e^{i\theta_k}$

where $\theta_k = \frac{\theta + 2\pi k}{m}$



$Z^3 = 1 \iff Z = \sqrt[3]{1}$

$w_0 = e^0$
$w_1 = e^{i\frac{2\pi}{3}}$
$w_2 = e^{i\frac{4\pi}{3}}$

$K = 0, 1, 2, \dots, m-1$

m من الجذور المختلفة تقع حول الدائرة بمسافات متساوية.

Example 5.2.2 The doubling map $D : [0,1] \to [0,1]$ is transitive: To show this we explicitly construct a point $x_0 \in [0,1]$ which has a dense orbit under D. x_0 is defined using its binary expansion in the following way: first write down all possible "1-blocks", i.e., 0 followed by 1. Then write down all possible "2-blocks", i.e., 00, 01, 10, 11, then all possible '3-blocks", i.e., 000, 001, 010, 011, 100, 101, 110, 111, and then continue in this way with all possible "4-blocks" etc. (we could write them down in the order in which they appear as in the binary expansion of the integers). This gives:

$$x_0 = \cdot 01\ 00\,01\,10\,11\ 000\,001\,010\,011\,100\,101\,110\,111\ 0000\,0001\ldots,$$

a point of $[0,1]$.

To show that $O(x_0)$ is dense in $[0,1]$, let $y \in [0,1]$ with binary expansion

$$y = .y_1y_2y_3\ldots = \sum_{i=1}^{\infty} \frac{y_i}{2^i}, \quad y_i = 0 \quad \text{or} \quad 1,$$

and let $\delta > 0$.

Choose N so large that $\frac{1}{2^N} < \delta$. All possible finite strings of 0's and 1's appear in the binary expansion of x_0, so the string $y_1y_2y_3\ldots y_N$ must also appear in the binary expansion of x_0.

It follows that for some $r \in \mathbb{Z}^+$ we have

نقطة عشوائية x_0

$$D^r(x_0) = \cdot y_1y_2y_3\ldots y_N b_{N+1}b_{N+2}\ldots, \quad \text{for some} \quad b_{N+1}, b_{N+2}, \ldots,$$

so that

$$|D^r(x_0) - y| = |\cdot y_1y_2\ldots y_N b_{N+1}b_{N+2}\ldots - \cdot y_1y_2\ldots y_N y_{N+1}y_{N+2}\ldots|$$

$$\leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^N} < \delta.$$

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This shows that any point $y \in [0,1]$ is arbitrarily close to the orbit of x_0 under D, so this orbit is dense in $[0,1]$ □

Remark 5.2.3 It is quite easy to see that for a transitive map $f : X \to X$ on a metric space, given any non-empty open sets U and V in X there exists $m \in \mathbb{Z}^+$ such that

$$U \cap f^m(V) \neq \emptyset.$$

Less easy to see is the converse of this statement, which holds for complete & separable metric spaces (X is *separable* if it has a countable dense subset - see Chapter 7 for the notion of completeness). This is the *Birkhoff Transitivity Theorem:*

any Cauchy seq is convergent

has countable dense subset.

(characterization to transitive function)

Theorem 5.2.4 *A continuous map f of a complete & separable metric space X is transitive if and only if for every pair U and V of non-empty open subsets of X there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$.*

Using this result it is easy to show directly that the angle doubling map is transitive, however, the proof is beyond the scope of this text.

5.3 The Definition of Chaos

Devaney was the first to define the notion of chaos, saying that a function is chaotic if it has a dense set of periodic points, it is transitive and also has what is called *sensitive dependence on initial conditions* (known as the "butterfly effect" in the popular literature). Subsequently it was shown that the first two requirements imply the third, so we define chaos as follows:

Definition 5.3.1 Let $f : X \to X$ be a map of the metric space X, then f is said to be *chaotic* if: (strongly chaotic) واحيانا تسمى

(i) The set of periodic points of f is dense in X.

(ii) f is transitive.

f 1-1 & onto & f^{-1}, f are continuous.

homeo & $f, f^{-1} \in C^1$ ($f \in C^1 \equiv f'$ exist and cont)

Examples 5.3.2 1. Homeomorphisms or diffeomorphisms of an interval $I \subseteq \mathbb{R}$ cannot be chaotic as they are never transitive. The same type of considerations apply to functions such as $\sin x$, $\cos x$, $\arctan x$ and the logistic map L_μ for $0 < \mu < 3$.

2. We have shown that the doubling map $D : [0, 1] \to [0, 1]$ has a dense set of periodic points and is transitive, so it is a chaotic map.

3. The tent map $T : [0, 1] \to [0, 1]$ is chaotic. Recall that if $x \in [0, 1]$ has a binary expansion $x = \cdot a_1\, a_2\, a_3 \ldots$, then $Tx = \begin{cases} \cdot a_2\, a_3\, a_4 \ldots & \text{if} \quad a_1 = 0 \\ \cdot a_2'\, a_3'\, a_4' \ldots & \text{if} \quad a_1 = 1, \end{cases}$ where $a_i' = 1$ if $a_i = 0$ and $a_i' = 0$ if $a_i = 1$. More generally we can see using an induction argument that ([47]):

$$T^n x = \begin{cases} \cdot a_{n+1}\, a_{n+2}\, a_{n+3} \ldots & \text{if} \quad a_n = 0 \\ \cdot a_{n+1}'\, a_{n+2}'\, a_{n+3} \ldots & \text{if} \quad a_n = 1. \end{cases}$$

We can use this to write down the periodic points of T. For example, the fixed points are $x = 0$ and $x = \cdot 1\,0\,1\,0 \ldots = 2/3$, and the period 2-points are

$$x_1 = \cdot 01\,10\,01\,10 \ldots = 2/5 \quad \text{and} \quad x_2 = \cdot 11\,00\,11\,00 \ldots = 4/5.$$

The period 3-points are

$$\cdot 010\,010\,010\,010\ldots = 2/7, \quad \cdot 100\,100\,100\,100\ldots = 4/7, \quad \cdot 110\,110\,110\,110\ldots = 6/7$$

and

$$\cdot 001\,110\,001\,110\ldots = 2/9, \quad \cdot 011\,100\,011\,100\ldots = 4/9, \quad \cdot 111\,000\,111\,000\ldots = 8/9.$$

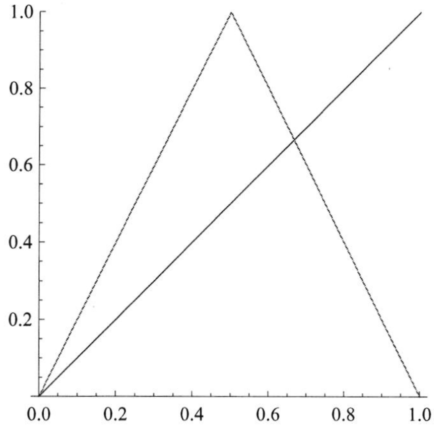
Notice that points of the form $x = k/2^n \in (0,1)$, $k \in \mathbb{Z}^+$, are almost fixed since they have a binary expansion of the form

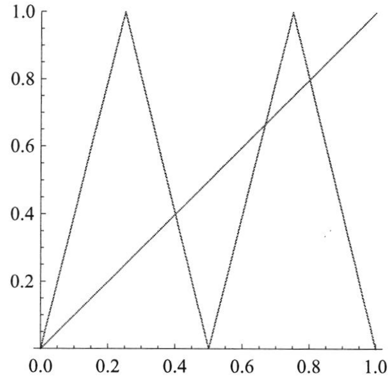
$$x = \cdot\, a_1\, a_2\, a_3 \ldots, a_n\, 0\, 0\, \ldots,$$

where $a_n = 0$ if k is even and $a_n = 1$ if k is odd. It follows that $T^n x = 0$ if k is even and $T^n x = 1$ if k is odd. In particular

$$T^n \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right] = [0,1].$$

The Intermediate Value Theorem now implies that there is a fixed point of T^n in the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}]$. Since such intervals can be made arbitrarily small and cover all of $[0,1]$, the set of periodic points must be dense in $[0,1]$.





The fixed and period 2-points of the tent map.

We use these ideas to show that the periodic points of T are numbers in $[0,1]$ of the form $x = r/s$ where r is an even integer and s is an odd integer.

Proof. If $x \in (0,1)$ is a periodic point for T, then $T^n x = x$ for some $n \in \mathbb{Z}^+$. There are two cases to consider: Suppose that x has a binary expansion

$$x = \cdot\, a_1\, a_2\, a_3 \ldots a_n\, a_{n+1} \ldots, \quad \text{where} \quad a_n = 0,$$

then

$$T^n x = \cdot a_{n+1}\, a_{n+2} \ldots a_{2n} \ldots,$$

so we must have $a_1 = a_{n+1}, a_2 = a_{n+2}, \ldots a_n = a_{2n} = 0$, and

$$x = \cdot a_1\, a_2 \ldots a_n\, a_1\, a_2 \ldots a_n\, a_1 \ldots, \quad \text{where} \quad a_n = 0,$$

Rewriting this gives

$$\begin{aligned} x &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{n-1}}{2^{n-1}}\right) + \frac{1}{2^n}\left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{n-1}}{2^{n-1}}\right) + \ldots \\ &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{n-1}}{2^{n-1}}\right)\left(\frac{1}{1 - 1/2^n}\right) \\ &= \frac{a_1 2^{n-1} + a_2 2^{n-2} + \ldots + a_{n-1} 2}{2^n - 1} = \frac{r}{s}, \end{aligned}$$

where r is even and s is odd.

In the second case $x = \cdot a_1\, a_2\, a_3 \ldots, a_n\, a_{n+1} \ldots$, where $a_n = 1$, so that

$$T^n x = \cdot a'_{n+1}\, a'_{n+2} \ldots a'_{2n} \ldots,$$

and this gives $a_1 = a'_{n+1},\ a_2 = a'_{n+2}, \ldots, a_n = a'_{2n}, \ldots$, and

$$x = \cdot a_1\, a_2 \ldots a_{n-1}\, 1\, a'_1\, a'_2 \ldots a'_{n-1}\, 0\, a_1 \ldots.$$

We can now argue as before, but using the first $2n$ terms of x.

Conversely, suppose that $x = r/s \in (0,1)$ where r is an even integer and s is an odd integer. Since s and 2 are coprime, we can apply Euler's generalization of Fermat's Theorem to give

$$2^{\phi(s)} \equiv 1 (\text{mod } s), \quad \text{or} \quad 2^p - 1 = ks \quad \text{for some} \quad p, k \in \mathbb{Z}^+,$$

$(1 < kr < 2^p - 1)$, where ϕ is Euler's function. Write the binary expansion of kr as

$$kr = a_1 2^{p-1} + a_2 2^{p-2} + \ldots a_{p-2} 2^2 + a_{p-1} 2, \quad a_i \in \{0,1\},$$

which is even, so

$$\begin{aligned} \frac{kr}{2^p - 1} &= (a_1 2^{p-1} + a_2 2^{p-2} + \ldots a_{p-2} 2^2 + a_{p-1} 2)\left(\frac{1/2^p}{1 - 1/2^p}\right) \\ &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{p-1}}{2^{p-1}}\right)\left(\frac{1}{1 - 1/2^p}\right) \\ &= \cdot a_1\, a_2 \ldots a_{p-1}\, 0\, a_1\, a_2 \ldots a_{p-1}\, 0 \ldots, \end{aligned}$$

which is a point of period p (or less). $\square$

Remark. $\boxed{\sum_{k=m+1}^{\infty} z^k = \frac{z^{m+1}}{1-z}, \ |z|<1.}$

$m+1=0$ نفس (2) لكن ← i.e. $m=-1,0,1,\cdots$

Proof. $\sum_{k=m+1}^{\infty} z^k = \sum_0^{\infty} z^k - \sum_0^{m} z^k = \frac{1}{1-z} - \frac{1-z^{m+1}}{1-z}, \ |z|<1$

$= \frac{1-1+z^{m+1}}{1-z} = \frac{z^{m+1}}{1-z}.$

الاثبات باستخدام

① $\sum_{k=0}^{m} z^k = \frac{1-z^{m+1}}{1-z}, \ z\neq 1$

② $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \ |z|<1$

It is now clear that T is transitive since if we define $x_0 \in (0,1)$ having a binary expansion consisting of all 1-blocks, all 2-blocks, all 3-blocks etc., as before, except that we insert a single zero between every block then it follows that if $x = \cdot x_1 x_2 \ldots x_n x_{n+1} \ldots$, then $T^p x_0 = \cdot x_1 x_2 \ldots x_n b_{n+1} \ldots$ for some $p > 0$, i.e., every block will appear in the iterates of x_0. As before we see that T is transitive, so is chaotic.

4. It is possible for a map to be transitive without being chaotic (although for continuous functions f on intervals in $\mathbb{R}$, this is not possible: see [50]). For example, consider the *irrational rotation* $R_a : S^1 \to S^1$ defined by $R_a(z) = a \cdot z$ for some (fixed) $a \in S^1$. To say that R_a is an irrational rotation means that $a^n \neq 1$ for any $n \in \mathbb{Z}^+$, i.e., a is not an nth root of unity for any $n \in \mathbb{Z}^+$. It can be shown that *every* $z_0 \in S^1$ has a dense orbit (a transformation with this property is said to be *minimal*). However, suppose that $R_a^n(z) = z$, then $a^n z = z$ or $a^n = 1$, a contradiction, so that R_a has no periodic points. R_a is an example of an *isometry*: points always stay the same distance apart:

$$|R_a(z) - R_a(w)| = |az - aw| = |a||z-w| = |z-w|.$$

Note that if instead we have $a^n = 1$ for some $n \in \mathbb{Z}^+$, then $R_a^n(z) = a^n z = z$ for all $z \in S^1$, so that R_a^n is just the *identity map* (every point of S^1 is of period n).

5.4 Some Symbolic Dynamics and the shift map

Recall that the set of all infinite sequences of 0's and 1's:

$$\Sigma = \{\omega = (s_1, s_2, s_3, \ldots) : s_i = 0 \text{ or } 1\},$$

is a metric space with metric defined by

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$$\boxed{d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k},} \quad \text{where} \quad \omega_1 = (s_1, s_2, \ldots), \ \ \omega_2 = (t_1, t_2, \ldots) \in \Sigma.$$

This metric has the following properties:

Property 5.4.1 If $\omega_1 = (s_1, s_2, \ldots)$, $\omega_2 = (t_1, t_2, \ldots) \in \Sigma$, with $s_i = t_i$, $i = 1, 2, \ldots, n$, then $d(\omega_1, \omega_2) \leq 1/2^n$.

متساويتان لـ n من الحدود (الاحداثيات) (coordinates) (من البداية)

Proof.

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k} = \sum_{k=n+1}^{\infty} \frac{|s_k - t_k|}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

□

(by) $\sum_{k=0}^{m} z^k = \frac{1-z^{m+1}}{1-z}, \ z\neq 1$

(by) $\sum_{0}^{\infty} z^k = \frac{1}{1-z}, \ |z|<1$

$\sum_{k=n+1}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2^k} - \sum_{k=0}^{n} \frac{1}{2^k}$

$= \frac{1}{1-\frac{1}{2}} - \frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}}$

$= 2 - 2\left(1 - \frac{1}{2^{n+1}}\right) = \frac{1}{2^n}.$

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Property 5.4.2 If $d(\omega_1, \omega_2) < 1/2^n$, then $s_i = t_i$ for $i = 1, 2, \ldots, n$.

Proof. We give a proof by contradiction: Suppose that $s_j = t_j$ for some $1 \leq j \leq n$, then

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k} \geq \frac{1}{2^j} \geq \frac{1}{2^n},$$

a contradiction. □

The shift map σ (sometimes called the *Bernoulli shift*) is an important function defined on Σ.

Definition 5.4.3 The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(s_1, s_2, s_3, \ldots) = (s_2, s_3, \ldots),$$

so for example $\sigma(1, 0, 1, 0, \ldots) = (0, 1, 0, 1, \ldots)$ and $\sigma^2(1, 0, 1, 0, \ldots) = (1, 0, 1, 0, \ldots)$, so that if $\omega_1 = (1, 0, 1, 0, \ldots)$ and $\omega_2 = (0, 1, 0, 1, \ldots)$, then $\{\omega_1, \omega_2\}$ is a 2-cycle for σ. In this way, it is easy to write down all of the points of period n. Any sequence which is eventually constant is clearly an eventually fixed point of Σ, and any sequence which is eventually periodic (such as $(1, 1, 1, 0, 1, 0, 1, 0, 1, \ldots)$), is an eventually periodic point.

Proposition 5.4.4 *The shift map $\sigma : \Sigma \to \Sigma$ is continuous, onto, but not one-to-one.*

infact it is two-to-one

Proof. Clearly σ is onto but not one-to-one. To show that σ is continuous, let $\epsilon > 0$, then we want to find $\delta > 0$ such that if

$$d(\omega_1, \omega_2) < \delta, \quad \text{then} \quad d(\sigma(\omega_1), \sigma(\omega_2)) < \epsilon.$$

We shall see that it suffices to take $\delta = 1/2^{n+1}$ if n is chosen so large that $1/2^n < \epsilon$.

In this case, if $d(\omega_1, \omega_2) < \delta = 1/2^{n+1}$, then from Property 5.4.2, $s_i = t_i$ for $i = 1, 2, \ldots, n+1$. Clearly the first n terms of the sequences $\sigma(\omega_1)$ and $\sigma(\omega_2)$ are equal, so by Property 5.4.1, $d(\sigma(\omega_1), \sigma(\omega_2)) \leq 1/2^n < \epsilon$, so that σ is continuous. □

We can now prove:

Theorem 5.4.5 *The shift map $\sigma : \Sigma \to \Sigma$ is chaotic.*

Proof. We first show that the periodic points are dense in Σ. Let $\boxed{\omega = (s_1, s_2, \ldots)} \in \Sigma$, then it suffices to show that there is a sequence of periodic points $\omega_n \in \Sigma$ with $\omega_n \to \omega$ as $n \to \infty$. Set

$\omega_1 = (s_1, s_1, s_1, s_1, \ldots)$, a period 1-point for σ,

$\omega_2 = (s_1, s_2, s_1, s_2, \ldots)$, a period 2-point for σ,

$\omega_3 = (s_1, s_2, s_3, s_1, s_2, s_3, \ldots)$, a period 3-point for σ,

and continuing in this way so that

$\omega_n = (s_1, s_2, \ldots, s_n, s_1, \ldots)$, a period n-point for σ. $= \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2}$ where $s_k = t_k$, $k = 1, \ldots, n$

Since ω and ω_n agree in the first n coordinates, $d(\omega, \omega_n) \leq 1/2^n$, so $d(\omega, \omega_n) \to 0$ as $n \to \infty$, or $\omega_n \to \omega$ as $n \to \infty$.

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2 To show that σ is transitive, we explicitly construct a point $\omega_0 \in \Sigma$ having a dense orbit under σ. Set

$$\omega_0 = (\underbrace{0\,1}_{\text{1-blocks}}\ \underbrace{00\,01\,10\,11}_{\substack{\text{all possible}\\ \text{2-blocks}}}\ \underbrace{000\,001\,010\,101\,011\ldots}_{\text{all possible 3 blocks}}),$$

and continuing in this way so that all possible n-blocks appear in ω_0. To see that $\overline{O(\omega_0)} = \Sigma$, let $\boxed{\omega = (s_1, s_2, s_3, \ldots)} \in \Sigma$ be arbitrary. Let $\epsilon > 0$ and choose n so large that $1/2^n < \epsilon$, then since ω_0 consists of all possible n-blocks, the sequence $(s_1, s_2, \ldots, s_n)$ must appear somewhere in ω_0, i.e., there exists $k > 0$ with

$$\sigma^k(\omega_0) = (s_1, s_2, \ldots, s_n, \ldots),$$

متطابقة

so that ω and $\sigma^k(\omega_0)$ agree on the first n coordinates. It follows that

متساوية (متوافقة متطابقة)

$$d(\omega, \sigma^k(\omega_0)) \leq \frac{1}{2^n} < \epsilon.$$

This shows that the orbit of ω_0 comes arbitrarily close to any member of Σ, so it is dense in Σ. It follows that σ is transitive and so it is chaotic. $\square$

Remark 5.4.6 It can be shown that (topological) properties such as being totally disconnected, perfect etc. are preserved by homeomorphisms.

The shift space Σ and the Cantor set C are homeomorphic metric spaces. In addition, the half open interval $[0, 1)$ and the unit circle S^1 in the complex plane are homeomorphic. In particular, Σ and C will have identical topological properties, and so will S^1 and $[0, 1)$. It follows that Σ and $[0, 1]$ cannot be homeomorphic as C and $[0, 1]$ are not homeomorphic (C is totally disconnected, but $[0, 1]$ is not).

Proof. Σ is given its usual metric, and C has the metric induced from being a subset of $\mathbb{R}$, so that $d(x, y) = |x - y|$ for $x, y \in C$.

We define a map $h : C \to \Sigma$ by $h(\cdot\, a_1\, a_2\, a_3 \ldots) = (s_1, s_2, s_3, \ldots)$, where $a_i = 0$ or 2 and $s_i = a_i/2$. Clearly h is both one-to-one and onto. We show that it is continuous at each $x_0 \in C$.

Let $\epsilon > 0$ and choose n so large that $1/2^n < \epsilon$. Set $\delta = 1/3^n$, then if $|x_0 - x| < \delta$, both x_0 and x must lie in the same component (sub-interval) of S_n of length $1/3^n$. It follows that x_0 and x must have an identical ternary expansions in the first n places. Correspondingly, $h(x_0)$ and $h(x)$ must have the same first n coordinates. It follows that $d(h(x_0), h(x)) \leq 1/2^n < \epsilon$, so h is continuous at x_0. In a similar way we see that h^{-1} is continuous.

To see that $[0, 1)$ and S^1 are homeomorphic, define $h : [0, 1) \to S^1$ by $h(x) = e^{2\pi i x}$, then h is clearly one-to-one and onto. The map h wraps the interval $[0, 1)$ around the circle, so that strictly speaking we are looking at $[0, 1]$ with the end points identified, so in this way h becomes continuous. □

Exercises 5.4

1. Let $f : X \to X$ be a transitive map of the metric space (X, d). Show that if U and V are non-empty open sets in X, there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$.

2. Let $F : [0, 1) \to [0, 1)$ be the tripling map $F(x) = 3x \bmod 1$. Follow the proof for the doubling map (but use ternary expansions) to show that F is transitive and the period points are dense (find the periodic points), and hence show that F is chaotic.

3. Let $D : [0, 1) \to [0, 1)$ be the doubling map. Show that $|\text{Per}_n(D)| = 2^n$.

cardinal no.

4. Use Proposition 4.4.7 to show that for a continuous increasing function $f : [a, b] \to [a, b]$, the periodic points cannot be dense in $[a, b]$ (so a homeomorphism of $[a, b]$ cannot be chaotic).

5.5 Sensitive Dependence on Initial Conditions

We now show that the original definition of chaos due to Devaney [12] follows from the definition we have given. This result is due to Banks et al. [4].

Δ for any δ>0

Definition 5.5.1 Let $f : X \to X$ be defined on a metric space (X, d). Then f has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any $x \in X$ and any open interval U containing x and points other than x, there is a point $y \in U$ and $n \in \mathbb{Z}^+$ with

$$d(f^n(x), f^n(y)) > \delta.$$

لأي نقطة x ~~x∈X~~ ~~ولأي δ>0~~ توجد نقطة قريبة قرباً كافياً من x بحيث مسارهما يكونان بعيدان بُعداً كافياً عن بعضهما عند زمن معين (n).

This is the precise definition of the idea that iterates of points close to each other may eventually be widely apart, so that a map has sensitive dependence on initial conditions if there exist points arbitrarily close to x which are eventually at least distance δ away from x. It is important to know whether we have sensitive dependence when doing computations as round-off errors may be magnified after numerous iterations. For example, suppose we iterate the doubling map, starting with $x_0 = 1/3$ and $x_1 = .333$. After 10 iterations we have $D^{10}(x_0) = 1/3$ and $D^{10}(x_1) = .92$, more than distance 1/2 apart.

Examples 5.5.2 1. The linear map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax$, $|a| > 1$, has sensitive dependence since if $x \neq y$,

$$|f^n(x) - f^n(y)| = |a^n x - a^n y| = a^n |x - y| \to \infty \quad \text{as} \quad n \to \infty.$$

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However, clearly the dynamics of f is not complicated (f is not chaotic). f not transitive

2. The shift map $\sigma : \Sigma \to \Sigma$ has sensitive dependence on initial conditions since if $\omega_1, \omega_2 \in \Sigma$ with $\omega_1 \neq \omega_2$, then they must differ at some coordinates, say $s_i \neq t_i$. Then

$$\sigma^{i-1}(\omega_1) = (s_i, s_{i+1}, \ldots), \quad \text{and} \quad \sigma^{i-1}(\omega_2) = (t_i, t_{i+1}, \ldots),$$

so that

$$d(\sigma^{i-1}(\omega_1), \sigma^{i-1}(\omega_2)) = \sum_{k=1}^{\infty} \frac{|s_{i+k-1} - t_{i+k-1}|}{2^k} = \frac{1}{2} + \text{other terms} \geq \frac{1}{2}.$$

3. The angle doubling map $f : S^1 \to S^1$, $f(z) = z^2$ has sensitive dependence since if we iterate $z = e^{i\theta}, w = e^{i\phi} \in S^1$, their distance apart doubles after each iteration.

4. The doubling map $D : [0, 1] \to [0, 1]$ can be seen directly to have sensitive dependence on initial conditions. This also follows from the following theorem:

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Theorem 5.5.3 (Banks et al. [4]) *Let $f : X \to X$ be a chaotic transformation, then f has sensitive dependence on initial conditions.*

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We first prove a preliminary result:

Lemma 5.5.4 *Let $f : X \longrightarrow X$ be a transformation which has at least two different periodic orbits. Then there exists $\epsilon > 0$ such that for any $x \in X$ there is a periodic point p satisfying*

$$d(x, f^k(p)) > \epsilon, \quad \textit{for all} \quad k \in \mathbb{Z}^+.$$

Proof. Let a and b be two periodic points with different orbits. Then $d(f^k(a), f^l(b)) > 0$ for all k and l (since we are dealing with finite sets).

Choose $\epsilon > 0$ small enough that $d(f^k(a), f^l(b)) > 2\epsilon$ for all k and l. Then

$$d(f^k(a), x) + d(x, f^l(b)) \geq d(f^k(a), f^l(b)) > 2\epsilon \quad \forall k, l \in \mathbb{Z}^+,$$

by the triangle inequality.

If x is within ϵ of any of the points $f^l(b)$, then it must be at a greater distance than ϵ from all of the points $f^k(a)$ and the result follows. $\square$

Proof of Theorem 5.5.3 Let $x \in X$ and U be an open set in X containing x.

The periodic points of f are dense in X, so there is a periodic point q of period r say:

$$q \in V = U \cap B_\delta(x),$$

Let p be a periodic point of period n for f, whose orbit is a distance greater than 4δ form x, and write

$$W_i = B_\delta(f^i(p)).$$

Now

$$f^i(p) \in W_i, \forall i \Rightarrow p \in f^{-i}(W_i), \forall i,$$

so the open set

$$W = f^{-1}(W_1) \cap f^{-2}(W_2) \cap \cdots \cap f^{-n}(W_n) \neq \emptyset.$$

Since f is transitive, there is a point $z \in V$ with $f^k(z) \in W$ for some $k \in \mathbb{Z}^+$.

Let j be the smallest integer with $k < nj$, or

$$1 \leq nj - k \leq n.$$

Then

$$f^{nj}(z) = f^{nj-k}(f^k(z)) \in f^{nj-k}(W).$$

But

$$f^{nj-k}(W) = f^{nj-k}(f^{-1}(W_1) \cap f^{-2}(W_2) \cap \cdots \cap f^{-n}(W_n)) \subset f^{nj-k}(f^{-(nj-k)}W_{nj-k} = W_{nj-k}),$$

so that $d(f^{nj}(z), f^{nj-k}(p)) < \delta$. Now $f^{nj}(q) = q$, so

$$\begin{aligned} d(f^{nj}(q), f^{nj}(z)) &= d(q, f^{nj}(z)) \\ &\geq d(x, f^{nj-k}(p)) - d(f^{nj-k}(p), f^{nj}(z)) - d(q, x), \end{aligned}$$

by the triangle inequality since

$$d(q, x) \leq d(x, f^{nj-k}(p)) - d(f^{nj-k}(p), f^{nj}(z)) + d(q, f^{nj}(z)).$$

So

$$d(f^{nj}(q), f^{nj}(z)) \geq 4\delta - \delta - \delta = 2\delta.$$

This inequality implies that either

$$d(f^{nj}(x), f^{nj}(z)) \geq \delta,$$

or

$$d(f^{nj}(x), f^{nj}(q)) \geq \delta,$$

for if $f^{nj}(x)$ were within distance $< \delta$ from both of these points, they would have to be within $< 2\delta$ from each other, contradicting the top inequality above. So one of the two, z or q will serve as the y in the theorem with $m = nj$. □

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conjugacy is transitive property

i.e. if $f \underset{h}{\approx} g$ and $g \underset{u}{\approx} K$ then $f \approx K$.

Proof. we have $h \circ f = g \circ h$ and — ①

$u \circ g = K \circ u$ — ②

T.P. $(u \circ h) \circ f \overset{?}{=} K \circ (u \circ h)$

and $u \circ h$ is homeomorphism,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] \arrow[dd, bend right, "u \circ h"'] & X \arrow[d, "h"] \arrow[dd, bend left, "u \circ h"] \\ Y \arrow[r, "g"] \arrow[d, "u"] & Y \arrow[d, "u"] \\ Z \arrow[r, "K"] & Z \end{tikzcd}$$

$(u \circ h) \circ f = u \circ (h \circ f)$

$= u \circ (g \circ h)$ by ①

$= (u \circ g) \circ h$

$= (K \circ u) \circ h$ by ②

$= K \circ (u \circ h)$, and since u and h are both homeo. then so as $(u \circ h)$.

$\therefore f \underset{u \circ h}{\approx} K$.

الترافق

Chapter 6. Conjugacy of Dynamical Systems

Two metric spaces X and Y are the "same" (homeomorphic) if there is a homeomorphism from one space to the other. In this chapter we study the question of when two *dynamical systems* are the same. Given maps $f : X \to X$ and $g : Y \to Y$, we require them to have the same type of dynamical behavior, e.g., if one is chaotic, then so is the other, there is a one-to-one correspondence between their periodic points etc. One obvious requirement is that the underlying metric spaces should be homeomorphic. We have seen a lot of similarities between the logistic map $L_4(x) = 4x(1-x)$ and the tent map $T(x)$ and this will be examined in this chapter together with other examples such as the shift map and circle maps.

6.1 Conjugate Maps الدوال المترافقة

This "sameness" is given by the idea of conjugacy, a notion borrowed from group theory, where two members a and b of a group G are conjugate if there exists $g \in G$ with $ag = gb$. One of the central problems of one-dimensional dynamics and dynamical systems in general is to be able to tell whether or not two dynamical systems are conjugate. We will see that if one map has a 3-cycle and another map has no 3-cycle (for example), then the maps cannot be conjugate, or if one map has 2 fixed points and the other has 3 fixed points, then they are not conjugate. These are examples of *conjugacy invariants*, which give criteria for maps to be non-conjugate. A generally harder question is establishing conjugacy between maps.

Definition 6.1.1 1. Let $f : X \to X$ and $g : Y \to Y$ be maps of metric spaces. Then f and g are said to be *conjugate* if there is a homeomorphism $h : X \to Y$ such that

$$\boxed{h \circ f = g \circ h.}$$

The map h is a *conjugacy* between f and g. Obviously conjugacy is an equivalence relation.

i.e. (1) $a \sim a$ reflexive
(2) $a \sim b \Rightarrow b \sim a$ symmetr
and (3) $a \sim b$ & $b \sim c \to a \sim c$ ← transitive relation.

2. If in the above definition we only require the map $h : X \to Y$ to be continuous, then we say that g is a *factor* of f. If in addition, h is an onto map, then we say that g is a *quasi-factor* of f.

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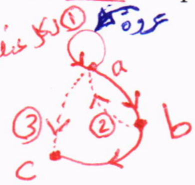

Exercises 6.1

1. Prove that if $f : X \to X$ and $g : Y \to Y$ are conjugate maps of metric spaces, then f is one-to-one if and only if g is one-to-one, and f is onto if and only if g is onto.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, "h \circ f"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

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the diagram Commute

$h \circ f = g \circ h \equiv f \underset{h}{\approx} g$

2. Prove that if f and g are conjugate via h and f has a local maximum at x_0, then g has a local maximum or minimum at $h(x_0)$.

3. Suppose that $h : [0,1] \to [0,1]$ is a conjugacy between $f, g : [0,1] \to [0,1]$ where $f(0) = f(1) = 0$ and $g(0) = g(1) = 0$. Show that h is increasing on $[0,1]$. Deduce that h maps the zero's of f to the zero's of g.

4. ([6]) The function $T_n(x) = \cos(n \arccos(x))$ is the nth Chebyshev polynomial. Show that T_n is conjuagte to the map $\Lambda_n : [0,1] \to [0,1]$, the piecewise linear continuous map defined by joining the points $(0,0), (1/n, 1), (2/n, 0), (3/n, 1), \ldots,$ ending with $(1,1)$ if n is odd, or $(1,0)$ if n is even. Use the conjugacy map $h : [0,1] \to [0,1]$, $h(x) = \cos(pix)$. (In [6], there is a generalization of this to maps T_λ, where $\lambda > 1$ is a real number).

6.2 Properties of Conjugate Maps

It is often easier to show that certain maps are chaotic indirectly by showing that they are conjugate to chaotic maps and using the following result:

Proposition 6.2.1 *If $f : X \to X$ and $g : Y \to Y$ are maps conjugate via a conjugacy $h : X \to Y$: $h \circ f = g \circ h$, then*

1. *$h \circ f^n = g^n \circ h$ for all $n \in \mathbb{Z}^+$, (so f^n and g^n are also conjugate).*

2. *If c is a point of period m for f, then $h(c)$ is a point of period m for g. c is attracting if and only if $h(c)$ is attracting.*

3. *f is transitive if and only if g is transitive.*

4. *f has a dense set of periodic points if and only if g has a dense set of periodic points.*

5. *f is chaotic if and only if g is chaotic.*

Proof. 1. $h \circ f^2 = h \circ f \circ f = g \circ h \circ f = g \circ g \circ h = g^2 \circ h$, and in the same way $h \circ f^3 = g^3 \circ h$, and continuing inductively the result follows.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, "h\circ f"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

T.P. $g^m(h(c)) = h(c)$ & $g^i \circ h(c) \neq h(c)$ $\forall i = 1, 2, \dots, m-1$

2. Suppose that $f^i(c) \neq c$ for $0 < i < m$ and $f^m(c) = c$, then $h \circ f^i(c) \neq h(c)$ for $0 < i < m$ since h is one-to-one, and so $g^i \circ h(c) \neq h(c)$ for $0 < i < m$. In addition, $h \circ f^m(c) = g^m \circ h(c)$, or $h(c) = g^m(h(c))$, so $h(c)$ is a period m point for g.

We shall only show that if p is an attracting fixed point of f (so that there is an open ball $B_\epsilon(p)$ such that if $x \in B_\epsilon(p)$ then $f^n(x) \to p$ as $n \to \infty$), then $h(p)$ is an attracting fixed point of g.

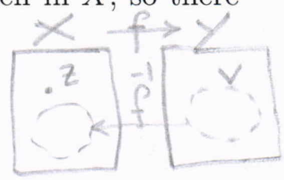
Let $V = h(B_\epsilon(p))$, then since h is a homeomorphism, V is open in Y and contains $h(p)$. Let $y \in V$, then $h^{-1}(y) \in B_\epsilon(p)$, so that $f^n(h^{-1}(y)) \to p$ as $n \to \infty$.

Since h is continuous, $h(f^n(h^{-1}(y))) \to h(p)$ as $n \to \infty$, i.e.,

$$g^n(y) = h \circ f^n \circ h^{-1}(y) \to h(p), \quad \text{as} \quad n \to \infty,$$

so $h(p)$ is attracting.

3. Suppose that $O(z) = \{z, f(z), f^2(z), \dots\}$ is dense in X and let $V \subset Y$ be a non-empty open set. Then since h is a homeomorphism, $h^{-1}(V)$ is open in X, so there exists $k \in \mathbb{Z}^+$ with $f^k(z) \in h^{-1}(V)$.



It follows that $h(f^k(z)) = g^k(h(z)) \in V$, so that

$$O(h(z)) = \{h(z), g(h(z)), g^2(h(z)), \dots\}$$

is dense in Y, i.e., g is transitive. Similarly, if g is transitive, then f is transitive.

4. Suppose that f has a dense set of periodic points and let $V \subset Y$ be non-empty and open. Then $h^{-1}(V)$ is open in X, so contains periodic points of f. As in (3), we see that V contains periodic points of g. Similarly if g has a dense set of periodic points, so does f.

5. This now follows from (3) and (4). $\square$

ستبقى صحيحة او ترابيا
يكفي quasi-factor ص 123

مثال على: صفة التحسس للشروط الابتدائية ليست صفة تنتقل بالدوال المترافقة (ليست صفة ترافقية)

Example 6.2.2 We remark that sensitive dependence on initial conditions is not a conjugacy invariant. It is possible for two maps on metric spaces to be conjugate, one to have sensitive dependence, but the other not: Consider $T : (0, \infty) \to (0, \infty)$, $T(x) = 2x$ and $S : \mathbb{R} \to \mathbb{R}$ defined by $S(x) = x + \ln 2$. If $H : (0, \infty) \to \mathbb{R}$ is defined by $\boxed{H(x) = \ln x}$, then H is a homeomorphism and we can check that $H \circ T = S \circ H$ so T and S are conjugate, T has sensitive dependence, but S does not.

It can be shown however, that if $T : X \to X$ is a map on a compact metric space X (for example $X = [0, 1]$) having sensitive dependence, then any map conjugate to T also has sensitive dependence.

$$\begin{tikzcd} (0,\infty) \arrow[r, "T"] \arrow[d, "H"'] & (0,\infty) \arrow[d, "H"] \\ \mathbb{R} \arrow[r, "S"'] & \mathbb{R} \end{tikzcd}$$

? ✓ (homeo.)

$H \circ T = S \circ H$?

let $x \in (0,\infty)$

① $H \circ T(x) = H(T(x)) = H(2x) = \ln 2x$

② $S \circ H(x) = S(H(x)) = S(\ln x) = \ln x + \ln 2 = \ln 2x$ ✓

It can also be shown that the property of having negative Schwarzian derivative is not a conjugacy invariant.

Example 6.2.3 1. The logistic map $L_4 : [0,1] \to [0,1]$, $L_4(x) = 4x(1-x)$ is conjugate to the tent map $T : [0,1] \to [0,1]$, $T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2(1-x) & \text{if } 1/2 < x \le 1 \end{cases}$.

Proof. Define $h : [0,1] \to [0,1]$ by $h(x) = \sin^2(\pi x/2)$. We can see that h is one-to-one, onto and both h and h^{-1} are continuous, so it is a homeomorphism (it is not a diffeomorphism as $h'(1) = 0$). Also

$$L_4 \circ h(x) = L_4\left(\sin^2(\frac{\pi x}{2})\right) = 4\sin^2(\frac{\pi x}{2})\left(1 - \sin^2(\frac{\pi x}{2})\right) = \sin^2(\pi x),$$

and

$$h \circ T(x) = h(Tx) = \begin{cases} h(2x) & \text{if } 0 \le x \le 1/2 \\ h(2-2x) & \text{if } 1/2 < x \le 1 \end{cases} = \sin^2(\pi x),$$

so $L_4 \circ h = h \circ T$ and L_4 and T are conjugate. ① $\sin 2\theta = 2\sin\theta\cos\theta$ نستخدم
② $\sin(\pi - \theta) = \sin\theta$ $\square$

2. The doubling map $D : [0,1] \to [0,1]$ is a quasi-factor of the shift map $\sigma : \Sigma \to \Sigma$.

Proof. D is a factor of σ since cont. & onto

$$h \circ \sigma = D \circ h,$$

where $h : \Sigma \to [0,1]$ is defined by

$$h(x_1, x_2, x_3, \ldots) = \cdot x_1\, x_2\, x_3 \ldots = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

is a continuous function (note that h is not a homeomorphism since it is not one-to-one: for example $h(1,0,0,\ldots) = 1/2 = h(0,1,1,1,\ldots)$, but $(1,0,0,\ldots) \neq (0,1,1,1,\ldots)$ in Σ). In addition $h(\Sigma) = [0,1]$, so that h is onto and D is a quasi-factor of σ $\square$

3 ~~4~~. The logistic map L_4 is a quasi-factor of the angle doubling map $f : S^1 \to S^1$, $f(z) = z^2$.

Proof. Define $h : S^1 \to [0,1]$ by $h(e^{ix}) = \sin^2 x$, then

$$L_4 \circ h(e^{ix}) = L_4(\sin^2 x) = 4\sin^2(1 - \sin^2 x) = \sin^2(2x),$$

and

$$h \circ f(e^{ix}) = h(e^{2ix}) = \sin^2(2x).$$

h is clearly onto and continuous, but it is not one-to-one: $h(e^{ix}) = h(e^{-ix})$, so L_4 is a quasi-factor of f, but h is not a conjugacy. $\square$

We can now show that many of the above maps are chaotic. In order to do this we need to weaken the conditions of Proposition 6.2.1. If we drop the requirement that h is a homeomorphism, but just require it to be continuous and onto, then we can show that if f is chaotic then so is g. In other words, if g is a quasi-factor of f where f is chaotic, then g is also chaotic. This result will be useful in showing that a number of well known examples are chaotic.

quasi-factor

Proposition 6.2.4 *Let $h : X \to Y$ be surjective (onto) and continuous. If $f : X \to X$ and $g : Y \to Y$ satisfy $h \circ f = g \circ h$ and f is chaotic, then g is chaotic.*

Before proving this proposition we need a lemma concerning continuous functions on metric spaces:

Lemma 6.2.5 *Let $h : X \to Y$ be a continuous function of metric spaces and A a subset of X, then $h(\overline{A}) \subseteq \overline{h(A)}$.*

Proof. Let $y \in h(\overline{A})$, then there exists $x \in \overline{A}$ with $y = h(x)$. We can find a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$.

Then $h(x_n) \in h(A)$ and since h is continuous

$$\lim_{n\to\infty} h(x_n) = h(x) = y \quad \text{so that} \quad y \in \overline{h(A)}.$$

$\square$

Proof of Proposition 6.2.4 Use $\text{Per}(f)$ and $\text{Per}(g)$ to denote the periodic points of f and g respectively, then we saw earlier that $h(\text{Per}(f)) \subseteq \text{Per}(g)$. Since f is chaotic, $\overline{\text{Per}(f)} = X$, and since h is onto, $h(X) = Y$. Then using the lemma we have

$$Y = h(X) = h(\overline{\text{Per}(f)}) \subseteq \overline{h(\text{Per}(f))} \subseteq \overline{\text{Per}(g)},$$

so that $\overline{\text{Per}(g)} = Y$. In other words, the periodic points of g are dense in Y.

f is transitive so there exists $x_0 \in X$ with $\overline{\text{O}_f(x_0)} = X$ (where we use the subscript to distinguish the orbits with respect to f and g). Now

$$h\left(\text{O}_f(x_0)\right) = h\{f^n(x_0) : n \in \mathbb{Z}^+\} = \{h \circ f^n(x_0) : n \in \mathbb{Z}^+\}$$

$$= \{g^n \circ h(x_0) : n \in \mathbb{Z}^+\} = \text{O}_g(h(x_0)),$$

so that

$$Y = h(X) = h(\overline{\text{O}_f(x_0)}) \subseteq \overline{h\left(\text{O}_f(x_0)\right)} = \overline{\text{O}_g(h(x_0))},$$

so that $h(x_0)$ is a transitive point for g. $\square$

It is easily seen that Proposition 6.~~3~~2.4 remains true if we replace the requirement that h be onto by requiring that $h(X)$ be dense in Y.

Theorem 6.2.6 *The tent map $T : [0,1] \longrightarrow [0,1]$, the logistic map $L_4(x) = 4x(1-x)$, the angle doubling map $f : S^1 \rightarrow S^1$, $f(z) = z^2$, and the Doubling map $D : [0,1] \rightarrow [0,1]$ are all chaotic.*

Proof. Suppose that we can show that the angle-doubling map f is a quasi-factor of the doubling map D, then we have:

The tent map T is conjugate to the logistic map L_4, which is a quasi-factor of $f(z)$, which is quasi-factor of D, which is a quasi-factor of σ, the shift map. It has been shown that the shift map is chaotic. The result now follows.

It therefore suffices to show that for $f : S^1 \rightarrow S^1$, $f(z) = z^2$ and $D : [0,1] \rightarrow [0,1]$, $D(x) = 2x \pmod 1$, f is a quasi-factor of D. Define $h : [0,1] \rightarrow S^1$ by $h(x) = e^{2\pi i x}$, then h is continuous, onto and one-to-one everywhere except that $h(0) = 1 = h(1)$. Now

$$f \circ h(x) = f(e^{2\pi i x}) = e^{4\pi i x},$$

and

$$h \circ D(x) = h(2x \pmod 1) = e^{2\pi i (2x \,(\text{mod} 1))} = e^{4\pi i x},$$

or $h \circ D = f \circ h$, and f is a quasi-factor of D. $\square$

Exercises 6.2

1. If $g(z) = z^3$ on S^1, show that g is the angle tripling map. Find the period points and show that they are dense in S^1. Show that g is conjugate to the map $F : [0,1) \rightarrow [0,1)$, $F(x) = 3x$ mod 1 of Exercises 5.4.

1. If $T_3 : C \rightarrow C$ is the tent map with $\mu = 3$, but restricted to the Cantor set. Show that T_3 is conjugate to the shift map $\sigma : \Sigma \rightarrow \Sigma$.

3. Prove that the doubling map $D : [0,1) \rightarrow [0,1)$, $D(x) = 2x \pmod 1$, and the angle doubling map $f : S^1 \rightarrow S^1$, $f(z) = z^2$, are conjugate.

4. Is the shift map $\sigma : \Sigma \to \Sigma$ conjugate to the doubling map D? (It can be shown that the shift map σ is conjugate to T_3 restricted to the Cantor set).

5. Let $U : [-1, 1] \to [-1, 1]$ be defined by $U(x) = 1 - 2x^2$ and $T_2 : [0, 1] \to [0, 1]$ be the tent map

$$T_2(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2(1-x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Prove that $H : [0, 1] \to [-1, 1]$, $H(x) = -\cos(\pi x)$ defines a conjugacy between these maps.

6. Prove that the map $F : [-1, 1] \to [-1, 1]$, $F(x) = 4x^3 - 3x$ is conjugate to $T : [0, 1] \to [0, 1]$, $T(x) = 3x \text{ mod}(1)$, via the conjugacy $h : [0, 1] \to [-1, 1]$, $h(x) = \cos(\pi x)$. (Hint: $\cos(3x) = 4\cos^3(x) - 3\cos(x)$). Deduce that F is chaotic. This question is related to Exercise 6.1, # 4 concerning the Chebyshev polynomials.

6.3 Linear Conjugacy

It is sometimes the case that the conjugacy between two real (or complex) functions is given by a map with a straight line graph (an affine map). This is called a linear conjugacy, and is stronger than the usual notion of conjugacy.

Definition 6.3.1 For functions $f : I \to I$ and $g : J \to J$ defined on subintervals of $\mathbb{R}$, we say that f and g are *linearly conjugate* and that h is a *linear conjugacy* if h maps I onto J where $h(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$ and $h \circ f = g \circ h$.

The following example gives a criterion for two quadratic functions to be linearly conjugate:

Example 6.3.2 Let

$$F(x) = ax^2 + bx + c \quad \text{and} \quad G(x) = rx^2 + sx + t,$$

where $a \neq 0$ and $r \neq 0$. If

$$1 \qquad c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a},$$

then F and G are linearly conjugate via the linear conjugacy

$$2 \quad h(x) = \frac{a}{r}x + \frac{b-s}{2r}.$$

Proof.

$$h \circ F(x) = h(ax^2 + bx + c) = \frac{a(ax^2 + bx + c)}{r} + \frac{b-s}{2r}$$
$$= \frac{a^2}{r}x^2 + \frac{ab}{r}x + \boxed{\frac{2ac + b - s}{2r}},$$

and

$$G \circ h(x) = G(\frac{a}{r}x + \frac{b-s}{2r}) = r(\frac{a}{r}x + \frac{b-s}{2r})^2 + s(\frac{a}{r}x + \frac{b-s}{2r}) + t$$
$$= r\left(\frac{a^2}{r^2}x^2 + 2\frac{a(b-s)}{2r^2}x + \frac{(b-s)^2}{4r^2}\right) + \frac{sa}{r}x + \frac{bs - s^2}{2r} + t$$
$$= \frac{a^2}{r}x^2 + \frac{ab}{r}x + \boxed{\frac{(b-s)^2 + 2bs - 2s^2 + 4rt}{4r}},$$

and we see that these are equal if

$$\boxed{c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a}.}$$

- إما نساويهما ونجد c
- أو نعوض عن c ونثبت
تساويهما.

□

For example, if F is defined on the interval $[0, 1]$, then

$$h(0) = \frac{b-s}{2r} \quad \text{and} \quad h(1) = \frac{2a + b - s}{2r},$$

so if $a/r > 0$, then F is conjugate to G on the interval $[\frac{b-s}{2r}, \frac{2a+b-s}{2r}]$.

Example 6.3.3 1. If $L_\mu(x) = \mu x(1-x)$, and $Q_c(x) = x^2 + c$ and $c = \dfrac{2\mu - \mu^2}{4}$, then L_μ on the interval $[0, 1]$ is linearly conjugate to Q_c on the interval $[-\mu/2, \mu/2]$. If $\mu = 4$, this shows that $L_4(x) = 4x(1-x)$ on $[0, 1]$ is conjugate to $Q_c(x) = x^2 + c$ on the interval $[-2, 2]$ when $c = -2$. In particular, Q_{-2} on $[-2, 2]$ is chaotic.

Proof. We apply Example 6.3.2 with

$$a = -\mu, \; b = \mu, \; c = 0, \; r = 1, \; s = 0, \; t = c.$$

In this case $h(0) = \mu/2$ and $h(1) = -\mu/2$ and we can check that the conditions of the example hold when $c = \dfrac{2\mu - \mu^2}{4}$. □

The fixed point Theorem. Any continuous map on a closed interval into itself has a fixed point, i.e, a point c s.t $f(c) = c$.
Proof. (define $g(x) = f(x) - x$ and use the intermediat value thm.)

2. On the other hand, if $\mu = 2$, we see that $L_2(x) = 2x(1-x)$ on $[0,1]$ is conjugate to $Q_0(x) = x^2$ on $[-1,1]$. Recall in Exercises 1.1 the difference equation $x_{n+1} = 2x_n(1-x_n)$ transforms to $y_{n+1} = y_n^2$ on setting $x_n = (1-y_n)/2$. This is just the fact that L_2 and Q_0 are conjugate via $h(x) = -2x+1$.

3. We can check that the logistic map L_4 is conjugate to $F : [-1,1] \to [-1,1]$, $F(x) = 2x^2 - 1$.

Exercises 6.3

1. Show that every quadratic polynomial $p(x) = ax^2 + bx + d$ is conjugate to a unique polynomial of the form $f_c(x) = x^2 + c$.

2. Prove that the logistic map L_4 is conjugate to $F : [-1,1] \to [-1,1]$, $F(x) = 2x^2 - 1$.
$L_4(x) = 4x - 4x^2$ $a = -4$, $b = 4$ $c = 0$ & $r = 2$, $s = 0$, $t = -1$
Since $c = 0 = \frac{1}{4a}(b^2 - s^2 + 2s - 2b + 4rt) = \frac{1}{16}(16 - 0 + 0 - 8 - 8) = 0$ So

3. Show that the logistic map $L_\mu(x) = \mu x(1-x)$, $x \in [0,1]$ is conjugate to the logistic type map $F_\mu(x) = (2-\mu)x(1-x)$ ($\mu \neq 2$), via the linear conjugacy (which is defined on the interval with end points $\frac{1-\mu}{2-\mu}$ and $\frac{1}{2-\mu}$):

$$h(x) = \frac{\mu}{2-\mu}x + \frac{1-\mu}{2-\mu}.$$

$c \in (0, \frac{1}{4}]$

4. Let $f_c(x) = x^2 + c$. Show that if $c > -1/4$, there is a unique $\mu > 0$ such that f_c is conjugate to $L_\mu(x) = \mu x(1-x)$ via a map of the form $h(x) = ax + b$.

5. (a) If $f_a(x) = ax$, $f_b(x) = bx$, $a, b \in \mathbb{R}$, defined on $\mathbb{R}$, when are they linearly conjugate?

(b) Show that $f_{1/2}$ and $f_{1/4}$ are conjugate via the map $h(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$.

للاطلاع

6.4 Conjugacy and the Tent Family

We saw in Section 2.7 that for $\mu \geq (1+\sqrt{5})/2$, the tent map T_μ has a point of period three, so by Sharkovsky's Theorem, it will have points of all possible periods. In this section we use a certain conjugacy to show that for $\mu > 1$, T_μ will have points of period 2^n for each $n \geq 1$. Our argument is based on that in [19]. We first show that the interval $[1/(1+\mu), \mu/(1+\mu)]$ is invariant under T_μ^2 when $1 < \mu \leq \sqrt{2}$.

The formula for T_μ^2 in Section 2.7 gives

$$T_\mu^2(x) = \begin{cases} \mu^2 x & \text{if } \ 0 \leq x \leq \frac{1}{2\mu} \\ \mu - \mu^2 x & \text{if } \ \frac{1}{2\mu} < x \leq \frac{1}{2} \\ \mu^2 x + \mu - \mu^2 & \text{if } \ \frac{1}{2} < x \leq 1 - \frac{1}{2\mu} \\ \mu^2 - \mu^2 x & \text{if } \ 1 - \frac{1}{2\mu} < x \leq 1 \end{cases}$$

Proposition 6.4.1 *For $1 < \mu \leq \sqrt{2}$, The restriction*

$$T_\mu^2 : [\frac{1}{1+\mu}, \frac{\mu}{1+\mu}] \to [\frac{1}{1+\mu}, \frac{\mu}{1+\mu}],$$

is well defined.

Proof. Note that for this range of values of μ, $1/(1+\mu) < 1/2$ and $\mu/(1+\mu) > 1/2$, so that $T_\mu(1/(1+\mu)) = \mu/(1+\mu)$ is an eventual fixed points since $T_\mu(\mu/(1+\mu)) = \mu/(1+\mu)$.

Let $x \in [1/(1+\mu), \mu/(1+\mu)]$, then from the formula for T_μ^2 and the fact that

$$\frac{1}{2\mu} < \frac{1}{1+\mu} < \frac{\mu}{1+\mu} < 1 - \frac{1}{2\mu},$$

we see that on this interval the minimum value of T_μ^2 occurs at $x = 1/2$ and this gives

$$T_\mu^2(x) \geq T_\mu^2(1/2) = \mu(1-\mu/2) \geq \frac{1}{1+\mu},$$

since this is equivalent to

$$\mu^3 - \mu^2 - 2\mu + 2 \leq 0, \quad \text{or} \quad (\mu - 1)(\mu^2 - 2) \leq 0,$$

where $1 < \mu \leq \sqrt{2}$.

Again assuming that $x \in [1/(1+\mu), \mu/(1+\mu)]$, we see that on the other hand, if $x \leq 1/2$, then $T_\mu(x) = \mu x > \mu/(1+\mu) > 1/2$, so $T_\mu^2(x) = \mu(1 - \mu x) < \mu(1 - \mu/(1+\mu)) = \mu/(1+\mu)$, so $T_\mu^2(x) \in [1/(1+\mu), \mu/(1+\mu)]$.

If instead $x > 1/2$, then

$$T_\mu(x) = \mu(1-x) > \mu(1 - \frac{\mu}{1+\mu}) = \frac{\mu}{1+\mu} > \frac{1}{2},$$

so

$$T_\mu^2(x) = \mu(1-\mu(1-x)) = \mu(1-\mu+\mu x) \leq \mu(1-\mu+\frac{\mu^2}{1+\mu}) = \frac{\mu}{1+\mu},$$

so again $T_\mu^2(x) \in [1/(1+\mu), \mu/(1+\mu)]$. □

We use the above proposition to show that T_μ and $T_{\sqrt{\mu}}^2$ are conjugate when $T_{\sqrt{\mu}}^2$ is restricted to a suitable invariant subinterval.

Proposition 6.4.2 *For $1 < \mu \leq \sqrt{2}$, T_μ^2 restricted to the interval*

$$\left[\frac{1}{1+\mu}, \frac{\mu}{1+\mu}\right],$$

is conjugate to T_{μ^2} on $[0,1]$.

Proof. From Proposition 6.4.1, we see that the given interval is invariant under T_μ^2. Now we show that we actually have a linear conjugacy $h(x) = ax + b$:

$$h \circ T_\mu^2 = T_{\mu^2} \circ h, ,$$

where

$$h : \left[\frac{1}{1+\mu}, \frac{\mu}{1+\mu}\right] \to [0,1],$$

and

$$a = \frac{1+\mu}{1-\mu}, \quad b = \frac{\mu}{\mu-1},$$

where we can check that

$$h(\frac{\mu}{1+\mu}) = 0, \quad \text{and} \quad h(\frac{1}{1+\mu}) = 1.$$

If $0 \leq x \leq 1/2$, then since $h^{-1}(x) = x/a - b/a$, we can check that $1/2 \leq h^{-1}(x) \leq \mu/(1+\mu) < 1 - 1/2\mu$, so that

$$h \circ T_\mu^2 \circ h^{-1}(x) = h \circ T_\mu^2 \left(\frac{x}{a} - \frac{b}{a}\right)$$

$$= h\left(\mu^2(\frac{x}{a} - \frac{b}{a}) + \mu - \mu^2\right) = \mu^2 x - \mu^2 b + a(\mu - \mu^2) + b = \mu^2 x = T_{\mu^2}(x).$$

Similarly we can check that if $1/2 < x \leq 1$, then $1/2\mu < 1/(1+\mu) \leq h^{-1}(x) \leq 1/2$, so that

$$h \circ T_\mu^2 \circ h^{-1}(x) = h \circ T_\mu^2(\frac{x}{a} - \frac{b}{a}) = h\left(\mu - \mu^2(\frac{x}{a} - \frac{b}{a})\right)$$

$$= a\mu - \mu^2 x + \mu^2 b + b = \mu^2(1-x) = T_{\mu^2}(x),$$

i.e., in both cases we have $h \circ T_\mu^2 \circ h^{-1}(x) = T_{\mu^2}(x)$, giving the desired conjugacy. $\square$

Corollary 6.4.3 *For $1 < \mu \leq 2$, $T^2_{\sqrt{\mu}}$ restricted to the interval*

$$\left[\frac{\sqrt{\mu}-1}{\mu-1}, \frac{\mu-\sqrt{\mu}}{\mu-1}\right],$$

is conjugate to T_μ on $[0,1]$.

We apply these results to give us information about the period points of T_μ:

Proof. Replace μ by $\sqrt{\mu}$ in the previous result. $\square$

Theorem 6.4.4 *For $1 < \mu \leq 2$, T_μ has a 2^n-cycle for each $n \in \mathbb{Z}^+$.*

Proof. We have seen that for each $\mu > 1$, T_μ has a period 2-point distinct from the fixed point of T_μ. In particular, as $\mu^2 > 1$, T_{μ^2} has a period 2-point distinct from the fixed point of T_{μ^2}. But by the last result, T_{μ^2} and T_μ^2 are conjugate, so T_μ^2 has a period 2-point distinct from the fixed point of T_μ^2. This must be a period 4-point for T_μ, for if not it would be a period 2-point, giving a fixed point for T_μ^2.

Continuing this argument, starting with a period 2-point for T_{μ^4} and the conjugacy between $T_{\mu^2}^2$ and T_{μ^4} we deduce that T_μ has a period 8-point. In this way, for each $n \in \mathbb{Z}^+$ we see that T_μ has a period 2^n-point. $\square$

Example 6.4.5 Consider the case where $\mu = 2$, then we see that T_2, the standard tent map, is conjugate to $T^2_{\sqrt{2}}$ when it is restricted to the interval $[\sqrt{2}-1, 2-\sqrt{2}]$. This implies that $T^2_{\sqrt{2}}$ has the same dynamics as T_2 on this subinterval. For example, it must have a 3-cycle, say $\{c_1, c_2, c_3\}$, where the c_i's are distinct and $T^6_{\sqrt{2}}(c_1) = c_1$. It follows that c_1 is a point of period 6 for $T_{\sqrt{2}}$, and in this way we deduce that $T_{\sqrt{2}}$ has $2k$-cycles for each $k \in \mathbb{Z}^+$. We saw earlier that $\mu = (1+\sqrt{5})/2$ is where period 3 is born for the tent family. In particular $T_{\sqrt{2}}$ has no 3-cycle, but if $\alpha = (1+\sqrt{5})/2$ and using the fact that $T^2_{\sqrt{\alpha}}$ (suitably restricted) is conjugate to T_α, it follows that $T_{\sqrt{\alpha}}$ must have points of period 6.

Remark 6.4.6 1. Suppose that $\mu > 1$ and $\dfrac{\mu^2}{1+\mu^3} \leq \dfrac{1}{2}$, then $\dfrac{\mu^3}{1+\mu^3} = 1 - \dfrac{1}{1+\mu^3} \geq \dfrac{1}{2}$, so that

$$T_\mu\left(\frac{\mu}{1+\mu^3}\right) = \frac{\mu^2}{1+\mu^3},\quad T_\mu\left(\frac{\mu^2}{1+\mu^3}\right) = \frac{\mu^3}{1+\mu^3} \quad \text{and} \quad T_\mu\left(\frac{\mu^3}{1+\mu^3}\right) = \frac{\mu}{1+\mu^3}.$$

We see that we have a 3-cycle:

$$\left\{\frac{\mu}{1+\mu^3}, \frac{\mu^2}{1+\mu^3}, \frac{\mu^3}{1+\mu^3}\right\}.$$

This 3-cycle appears when $\mu > 1$ and $\dfrac{\mu^2}{1+\mu^3} \leq \dfrac{1}{2}$, or equivalently

$$\mu^3 - 2\mu^2 + 1 \geq 0, \quad \text{or} \quad (\mu - 1)(\mu^2 - \mu - 1) \geq 0.$$

We see that this will happen when $\mu \geq (1+\sqrt{5})/2$. A similar analysis can be done for other periodic orbits. For example, if $\mu > 1$ and $\dfrac{\mu^3}{1+\mu^4} \leq \dfrac{1}{2}$ we get a 4-cycle, and this happens when $\mu^3 - \mu^2 - \mu - 1 \geq 0$.

2. Suppose that $1 < \mu < 2$, then if $x \in [\mu - \mu^2/2, \mu/2]$, we can check that $T_\mu(x) \in [\mu - \mu^2/2, \mu/2]$, so that this interval is an invariant set. For $1 < \mu < \sqrt{2}$, the smallest set invariant under T_μ is a collection of subintervals of $[\mu - \mu^2/2, \mu/2]$. If $\mu > \sqrt{2}$ this becomes all of the interval $[\mu - \mu^2/2, \mu/2]$, called the *Julia set* of T_μ (named after one of the early pioneers of chaotic dynamics, Gaston Julia, who worked on complex dynamics in particular in the early 1900's). For $\mu = 2$, the Julia set is all of $[0, 1]$. The bifurcation diagram for T_μ, $\mu > 1$ gives us some insight into what is happening here.

3. The conjugacy between T_2 and L_4 can be constructed by consideration of the periodic points of these maps. Since the period points are dense for each of these maps, by carefully ordering them according to their ordering in $[0, 1]$, we can define a map h by defining it on the periodic points. h is then defined on a dense subset of $[0, 1]$, into a dense subset. This map can be continuously extended to a homeomorphism of $[0, 1]$ with $h(0) = 0, h(1) = 1$. In this way it can be shown that the conjugation between T_2 and L_4 is unique. See the exercises for a proof that the conjugation between T_2 and L_4 is unique.

6.5 Renormalization

The previous example shows that T_μ^2 restricted to the interval

$$\left[\frac{\mu-1}{\mu^2-1}, \frac{\mu^2-\mu}{\mu^2-1}\right] = \left[\frac{1}{\mu+1}, \frac{\mu}{\mu+1}\right],$$

is conjugate to T_{μ^2} on $[0,1]$ (where we are replacing μ by μ^2 in 6.4.2). How do we arrive at this conjugacy? Notice that for $\mu > 1$, T_μ has a fixed point $p_\mu = \mu/(\mu+1)$ and another point $\hat{p}_\mu = 1/(\mu+1)$ with $T_\mu(\hat{p}_\mu) = T_\mu(p_\mu)$, so it is eventually fixed. Let us look at the graph of T_μ^2 restricted to the interval $[\hat{p}_\mu, p_\mu]$. Inside the square shown we see that the graph obtained looks like an "upside-down" version of T_μ, and we consider the possibility that T_μ^2 restricted to the interval $[\hat{p}_\mu, p_\mu]$ is actually conjugate to T_μ (or in fact T_{μ^2}).

Define a linear map $h_\mu : [\hat{p}_\mu, p_\mu] \to [0,1]$ of the form $h_\mu(x) = ax + b$ in such a way that $h_\mu(p_\mu) = 0$ and $h_\mu(\hat{p}_\mu) = 1$. We can check that

$$h_\mu(x) = \frac{1}{\hat{p}_\mu - p_\mu}(x - p_\mu), \quad \text{and} \quad h_\mu^{-1}(x) = (\hat{p}_\mu - p_\mu)x + p_\mu.$$

h_μ expands the interval $[\hat{p}_\mu, p_\mu]$ onto the interval $[0,1]$ and changes the orientation. This is exactly the conjugacy defined in Proposition 6.4.2.

We define a *renormalization operator* of T_μ by

$$(RT_\mu)(x) = h_\mu \circ T_\mu^2 \circ h_\mu^{-1}(x).$$

What we actually showed in the previous section is that $(RT_\mu)(x) = T_{\mu^2}(x)$, giving us the conjugacy claimed. This procedure can be continued for T_μ^4, T_μ^8 etc, and similar considerations can be made with the logistic map L_μ (see [12] for more details).

6.6 Conjugacy and Fundamental Domains

We have seen that two dynamical systems f and g with different dynamical properties cannot be conjugate. On the other hand, sometimes we have dynamical systems having seemingly very similar dynamical properties and which we would like to show are conjugate. This is sometimes possible using the notion of *fundamental domain*, a set on which we construct a map h in an arbitrary manner and show that it extends to a conjugacy on the whole space. We first illustrate this idea with homeomorphisms $f, g : \mathbb{R} \to \mathbb{R}$. We look at a fairly straightforward case where both homeomorphisms are order preserving and have no fixed points (in fact lie strictly above the line $y = x$).

Proposition 6.6.1 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be homeomorphisms satisfying $f(x) > x$ and $g(x) > x$ for all $x \in \mathbb{R}$. Then f and g are conjugate.*

Proof. The idea for the proof is a follows: Select $x_0 \in \mathbb{R}$ arbitrarily and consider the *2-sided orbit*

$$O_f(x_0) = \{f^n(x_0) : n \in \mathbb{Z}\} = \{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}.$$

Since $f(x) > x$ for all x, this is an increasing sequence: $\ldots x_{-1} < x_0 < x_1 < x_2 < \ldots$, so that the sets

$$\ldots, [x_{-1}, x_0), [x_0, x_1), [x_1, x_2), \ldots,$$

are disjoint and their union is all of $\mathbb{R}$. We must have $\lim_{n\to\infty} x_n = \infty$ since otherwise the limit would exist and would have to be a fixed point. There are no fixed points since $f(x) > x$ always. لأنّ الاقتراب فقط للنقطة الثابتة

The set $I = [x_0, f(x_0)) = [x_0, x_1)$ is called a *fundamental domain* for f. Set $J = [x_0, g(x_0))$ and define a map $h : I \to J$ arbitrarily as a continuous bijection (e.g., we can set $h(x_0) = x_0$ and $h(f(x_0)) = g(x_0)$ and then linearly from I to J).

Now every other orbit of f intertwines with $O_f(x_0)$: if $y_0 \in (x_0, x_1)$, then $y_n = f^n(y_0) \in f^n(I)$, so lies between x_i and x_{i+1}. It follows that every orbit has a unique member in the interval $[x_0, x_1)$ and we use this to extend the definition of h to all of $\mathbb{R}$.

If $x \in f^n(I)$ we define $h(x)$ by mapping x back to I via f^{-n}, then using $h(f^{-n}(x))$ which is well defined, and then mapping back to $g^n(J)$ using g^n. I.e., if $x \in f^n(I)$, $n \in \mathbb{Z}$, define

$$h(x) = g^n \circ h \circ f^{-n}(x).$$

In this way, h is defined on all of $\mathbb{R}$. We can check that h is one-to-one. It is onto because $h(f^n(I)) = g^n(J))$ for each n, and we can check that it is continuous. Finally, because of the definition of h, if $x \in \mathbb{R}$, then $x \in f^n(I)$ for some $n \in \mathbb{Z}$, so $x = f^n(y)$ for some $y \in I$. Then

$$g \circ h(x) = g(g^n \circ h \circ f^{-n}(x)) = g^{n+1} \circ h \circ f^{-(n+1)}(f(x)) = h \circ f(x),$$

so that f and g are conjugate. $\square$

Examples 6.6.2 1. The above argument can be generalized to the situation where $f(x)$ and $g(x)$ are homeomorphisms with corresponding fixed points. To be more precise, consider a homeomorphism $f : [0, 1] \to [0, 1]$ which is orientation preserving, so that $f(0) = 0$, $f(1) = 1$ and f is increasing. Suppose that f has fixed points (in addition to 0 and 1), at $c_1, c_2, \ldots, c_n$, then f^2 has the same collection of fixed points (no additional fixed points as f cannot have points of period 2 or higher). If $f(x) > x$ for $c_k < x < c_{k+1}$, then we can use the argument of the proposition to construct a homeomorphism between f and f^2, and do the same for each interval

$[c_i, c_{i+1}]$ (treating the case where $f(x) < x$ in an analogous way). In this way we see that f and f^2 are conjugate maps.

2. Consider the logistic maps $L_\mu(x) = \mu x(1-x)$ for various values of $\mu \in (0,4]$ and $x \in [0,1]$. We first show that for $0 < \mu < \lambda \leq 1$, L_μ and L_λ are conjugate. There is a slight complication here as these maps are not increasing, but they do have an attracting fixed point at 0, and we saw earlier that the basin of attraction is all of $[0,1]$. We first deal with the interval on which the maps are increasing, $[0,1/2]$, and look at the restriction of the functions to this interval.

Our aim is to construct a homeomorphism $h : [0,1] \to [0,1]$ with the property $L_\lambda \circ h = h \circ L_\mu$. Take $(L_\mu(1/2), 1/2] = (\mu/4, 1/2]$ as a fundamental domain for L_μ and $(L_\lambda(1/2), 1/2] = (\lambda/4, 1/2]$ as a fundamental domain for L_λ. Define $h : (\mu/4, 1/2] \to (\lambda/4, 1/2]$ by $h(1/2) = 1/2$ and $h(\mu/4) = \lambda/4$ and then linearly on the remainder of the interval.

Set $I = (\mu/4, 1/2]$ and $J = (\lambda/4, 1/2]$, then since 0 is an attracting fixed point, the intervals $L_\mu^n(I)$ and $L_\lambda^n(J)$ are disjoint for $n \in \mathbb{Z}^+$, and their union is all of $(0,1/2]$. Extend the definition of h so that it is defined on $(0,1/2]$ by;

$$h(x) = L_\lambda^n \circ h \circ L_\mu^{-n}(x), \quad \text{for} \quad x \in L_\mu^n(I).$$

We can now check that h is continuous and increasing on $[0,1/2]$ when we set $h(0) = 0$.

Now define h on $(1/2,1]$ by setting $h(1-x) = 1 - h(x)$ for $x \in [0,1/2)$, clearly giving a homeomorphism on $[0,1]$. Then

$$L_\lambda(h(1-x)) = L_\lambda(1-h(x)) = L_\lambda(h(x)) = h(L_\mu(x)) = h(L_\mu(1-x)),$$

so that h is the required conjugation. $\square$

3. A similar proof shows that L_μ and L_λ are conjugate whenever $1 < \mu < \lambda < 2$. Look at the intervals $[0, 1-1/\mu]$ and $[1-1/\mu, 1/2]$ separately and the fact that $1-1/\mu$ is an attracting fixed point, then use the symmetry about the point $x = 1/2$.

However, these maps cannot be conjugate to L_2 since any conjugating map $h : [0,1] \to [0,1]$ must have the property that $h(1/2) = 1/2$ (see the exercises). This leads to a contradiction.

4. The maps L_4 and L_μ, $\mu \in (0,4)$ cannot be conjugate since $L_4 : [0,1] \to [0,1]$ is an onto map, but L_μ is not (see the exercises).

Exercises 6.6

1. (a) Let $a, b \in (0,1)$ and $f_a(x) = ax$, $f_b(x) = bx$ be dynjamical systems on $[0,1]$. We saw in Exercises 6.3 that these maps need not be linearly conjugate. Prove that f_a and f_b are conjugate (Hint: Use the method of examples in this section).

(b) Let $g : [0,1] \longrightarrow [0,1]$ be continuous, strictly increasing with $g(0) = 0$ and $g(x) < x$ for all $x \in (0,1]$. Prove that g is conjugate to f_a for any $a \in (0,1)$.

2. Consider the tent map $T_{\sqrt{2}}$.

(i) Show that $x = 1/2$ is an eventual fixed point for $T_{\sqrt{2}}$.

(ii) Use Section 6.4 to show that there is a subinterval of $[0,1]$ on which $T_{\sqrt{2}}$ is conjugate to T_2.

(iii) Deduce that $T_{\sqrt{2}}$ has periodic points of period $2k$ for any $k > 1$, but no points of odd period greater than 1.

(iv) Prove that if $\mu > \sqrt{2}$, then T_μ has a 3-cycle.

(v) Prove that there is an interval on which $T_{\sqrt{2}}$ is chaotic.

3. Let $0 < \lambda, \mu < 1$. If $h : [0,1] \longrightarrow [0,1]$ is an orientation preserving homeomorphism with $h \circ L_\mu(x) = L_\lambda \circ h(x)$ for all $x \in [0,1]$, show that $h(1/2) = 1/2$ (Hint: h is a conjugation between two different logistic maps with $h \circ L_\mu(x) = L_\lambda \circ h(x)$. Note that this equation also holds if we replace x by $1 - x$. Use this to deduce that $h(x) + h(1-x) = 1$ for all $x \in [0,1]$).

4. Use exercise 2 above to deduce that $h(\mu/4) = \lambda/4$. It follows that the orientation preserving homeomorphism of Example 6.6.2 can be extended in only one way from $[\mu/4, 1/2]$ to $[\lambda/4, 1/2]$.

5*. Prove that for the logistic map L_μ, if $0 < \mu \leq 2$, then L_μ is conjugate to L_μ^2, the composition of L_μ with itself. Show that this is not true for $\mu > 2$. What is the corresponding result for the tent family?

6. Let $S_\mu(x) = \mu \sin(x)$. Prove that if $0 < \mu < \lambda < 1$, then S_μ and S_λ are conjugate.

7. Prove that the rotation $R_a : S^1 \to S^1$, $R_a(z) = az$ is conjugate to the map $T_\alpha : [0,1) \to [0,1)$, $T_\alpha(x) = x + \alpha \pmod 1$, when $a = e^{2\pi i\alpha}$. Can R_a be conjugate to R_b for $a \neq b$?

8. Prove that $T_\alpha : [0,1) \to [0,1)$, $T_\alpha(x) = x + \alpha \pmod 1$ is conjugate to its inverse map $T_\alpha^{-1}(x) = x - \alpha \pmod 1$. Can T_α be conjugate to T_α^2?

9. The aim of this exercise is to show the uniqueness of the conjugacy between the tent map T_2 and the logistic map L_4.

(i) Check that this conjugacy $k : [0,1] \to [0,1]$ is given by,

$$k(x) = \frac{2}{\pi}\arcsin(\sqrt{x}); \quad T_2 \circ k = k \circ L_4.$$

(ii) Suppose that $h : [0,1] \to [0,1]$ is another conjugacy between T_2 and L_4, then $h(0) = 0$, $h(1) = 1$ and h is a strictly increasing continuous function (why?), i.e., h is an orientation preserving homeomorphism of $[0,1]$.

(iii) Show that h maps the local maxima (respectively minima) of T_2^n to local maxima (respectively minima) of L_4^n.

(iv) Use the fact that any such conjugation is order preserving to show that $h(x) = k(x)$ at all local maxima and local minima.

(v) Use the continuity of h and k to deduce that $h(x) = k(x)$ for all $x \in [0,1]$.

(vi) Deduce that there is no C^1 conjugacy between T_2 and L_4.

10. Use the above exercise to show that if $L_4(x) = 4x(1-x)$, then L_4^n has turning points at $\sin^2(k\pi/2^{n+1})$, for $k = 1, 2, \ldots, 2^{n+1} - 1$.

11. Use the fact that the conjugation between T_2 and L_4 is unique to show that if $\phi : [0,1] \to [0,1]$ is a homeomorphism satisfying $L_4 \circ \phi = \phi \circ L_4$, then $\phi(x) = x$ for all $x \in [0,1]$, i.e., ϕ is the identity map (hint: first show that $k \circ \phi$ is also a conjugation between T_2 and L_4).

12*. Let T_μ be the tent map. Show that if $\mu > \sqrt{2}$, then for each open interval $U \subset [0,1]$, there exists $n > 0$ such that

$$\left[T_\mu^2(1/2), T_\mu(1/2)\right] \subseteq T_\mu^n(U).$$

(Hint: Use the fact that $|T_\mu(U)| \geq \mu|U|$ if U does not contain $1/2$, so that the length keeps increasing. We claim there exists $m > 0$ such that $T^m(U)$ and $T^{m+1}(U)$ both contain $1/2$, for if not, $|T^{m+2}(U)| \geq \mu^2|U|/2$ for all $m \in \mathbb{Z}^+$, a contradiction, since this eventually exceeds 1).

We shall now show that the logistic map $L_\mu : [0,1] \to [0,1]$, $L_\mu(x) = \mu x(1-x)$ with $0 < \mu < 4$ has at most one attracting cycle. We use a result due to Singer (1978) which is applicable to maps having a negative Schwarzian derivative. Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is C^3 if $f'''(x)$ exists and is continuous.

7.1 The Schwarzian Derivative Revisited

Recall the Schwarzian derivative of $f(x)$ is:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x))}\right]^2 = F'(x) - \frac{1}{2}\left[F(x)\right]^2,$$

where $F(x) = \dfrac{f''(x)}{f'(x)}$.

Our first goal is to show that many polynomials have negative Schwarzian derivatives.

Lemma 7.1.1 *Let f(x) be a polynomial of degree n for which all the roots of its derivative $f'(x)$ are distinct and real. Then $Sf(x) < 0$ for all x.*

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Proof. Suppose that the derivative of $f(x)$ is given by

$$f'(x) = a(x - r_1)(x - r_2) \cdots (x - r_{n-1}),$$

where $a \in \mathbb{R}$, then

$$F(x) = \frac{f''(x)}{f'(x)} = (\ln f'(x))' = \sum_{i=1}^{n-1} \frac{1}{x - r_i},$$

and so

$$F'(x) = -\sum_{i=1}^{n-1} \frac{1}{(x - r_i)^2}.$$

Now put this into the Schwarzian derivative:

$$Sf(x) = F'(x) - \frac{1}{2}[F(x)]^2 < 0.$$

$\square$

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Lemma 7.1.2 *Assume that f is a C^3 map on $\mathbb{R}$, then*

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(i) $S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x)$.

(ii) *If $Sf < 0$ and $Sg < 0$, then $S(f \circ g) < 0$.*

(iii) *If $Sf < 0$, then $Sf^k < 0$ for all $k \in \mathbb{Z}^+$.*

Proof. (i) As above we have $F(x) = \dfrac{f''(x)}{f'(x)}$, so set $G(x) = \dfrac{g''(x)}{g'(x)}$ and $H(x) = \dfrac{h''(x)}{h'(x)}$, where $h = f \circ g$. Then

$$h'(x) = f'(g(x)) \cdot g'(x), \quad h''(x) = (f''(g(x))g'(x)^2 + (f'(g(x))g''(x),$$

so that

$$\begin{aligned} H(x) &= \frac{f''(g(x))g'(x)^2 + f'(g(x))g''(x))}{f'(g(x))g'(x)} \\ &= \frac{f''(g(x))g'(x)}{f'(g(x))} + \frac{g''(x)}{g'(x)} = F(g(x))g'(x) + G(x). \end{aligned}$$

This gives

$$H'(x) = F'(g(x))g'(x)^2 + F(g(x))g''(x) + G'(x),$$

so that

$$S(f \circ g)(x) = H'(x) - \frac{1}{2}H^2(x)$$

$$= \left[F'(g(x)) - \frac{1}{2}F^2(g(x))\right] g'(x)^2 + F(g(x))g''(x) - F(g(x))g'(x)G(x) + G'(x) - \frac{1}{2}G^2(x)$$

$$= Sf(g(x)) \cdot g'(x)^2 + Sg(x),$$

since $G(x) = g''(x)/g'(x)$.

(ii) is now immediate, and (iii) follows using induction. $\square$

Example 7.1.3 Let $g(x) = \dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$, a linear fractional transformation. A direct calculation shows that $Sg(x) = 0$ everywhere in its domain. It now follows from Lemma 1 that if $h(x) = g(f(x)) = \dfrac{af(x)+b}{cf(x)+d}$. then $Sh(x) = Sf(x)$.

We now prove a version of Singer's Theorem. Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $c \in \mathbb{R}$ is an attracting fixed point or attracting periodic point, then the basin of attraction $B_f(c)$ is an open set. Denote by W the maximal open interval contained in $B_f(c)$ which contains c (called the *immediate basin of attraction of* c).

Theorem 7.1.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 map with negative Schwarzian derivative. If c is an attracting periodic point for f, then either:*

(i) *the immediate basin of attraction of c extends to ∞ or $-\infty$, or*

(ii) *there is a critical point of f (i.e., a root of $f'(x) = 0$), whose orbit is attracted to the orbit of c under f.*

Proof. We first look at the case where c is a fixed point of f. Suppose that its immediate basin of attraction is the open interval W and that (i) does not hold, then W is a bounded set.

Thus $W = (a, b)$ for some $a, b \in \mathbb{R}$. Because of the continuity of f and the fact that $f(a), f(b) \notin (a, b)$, there are three possibilities for $f(a)$ and $f(b)$.

Case 1: $f(a) = f(b)$ - this will happen for example if a is a fixed point and b is an eventual fixed point. It follows from the Mean Value Theorem that (a, b) contains a critical point of f.

Case 2: $f(a) = a$ and $f(b) = b$, then by the Mean Value Theorem, there are points $x_1 \in (a, c)$ and $x_2 \in (c, b)$ such that $f'(x_1) = f'(x_2) = 1$ (see picture). But since c is an attracting fixed point, $|f'(c)| \leq 1$. It follows that either $f'(x_0) = 0$ for some $x_0 \in (x_1, x_2)$, or $f'(x)$ has a minimum value $f'(x_0) > 0$. In the latter case we have $f'(x_0) > 0$, $f''(x_0) = 0$ and $f'''(x_0) > 0$, so that $Sf(x_0) > 0$, contradicting the Schwarzian derivative being everywhere negative.

Case 3: $f(a) = b$ and $f(b) = a$. Here c is fixed by f^2 and so are a and b, so that $(f^2)'$ has a zero x_0 in (a, b) (as in Case 1). But

$$(f^2)'(x_0) = (f'(f(x_0))f'(x_0) = 0,$$

so either x_0 or $f(x_0)$ is a root of f', but both lie in (a, b).

Now suppose that c is a point of period k, then $f^k(c) = c$, an attracting fixed point for f^k, then from our earlier arguments, the immediate basin of attraction of c (for f^k) contains a critical point of f^k, say x_0:

$$(f^k)'(x_0) = f'((x_0)f'(f(x_0)) \cdots f'(f^{k-1}(x_0)) = 0,$$

so that $f'(f^m(x_0)) = 0$ for some $0 \leq m < k$. In this case $f^m(x_0) \in f^m(W) \subset W$, the basin of attraction of c. $\square$

Example 7.1.5 Consider the map $f(x) = x - x^5$. We see that $f'(x)$ has two real roots: $\pm(1/5)^{1/4}$, and $f''(x)$ and $f'''(x)$ are both continuous. We have $f''(x) = -20x^3$, and $f'''(x) = -60x^2$.

Substituting these into the Schwarzian derivative gives:

$$Sf(x) = (-60x^2)/(1-5x^4) - 3/2[(-20x^3)/(1-5x^4)]^2 = \frac{-60x^2}{(1-5x^4)^2}(1+5x^4),$$

and this is always negative. We can check that the critical points are in the basin of attraction of the fixed point $x = 0$.

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Example 7.1.6 The map $f(x) = 3x/4 + x^3$ cannot have a negative Schwarzian derivative. It has fixed points $x = 0$ and $x = \pm 1/2$, 0 being attracting with bounded basin of attraction, but f has no critical points.

7.2 Singer's Theorem

Corollary 7.2.1 *Let $f : [0,1] \to [0,1]$ be a C^3 map with $Sf(x) < 0$ for all x. The basin of attraction of an attracting cycle contains 0, 1 or a critical point of $f(x)$.*

Proof. If $J = (a,b)$, $0 < a < b < 1$, is the basin of attraction of an attracting cycle, then we have seen above that it must contain a critical point of f. Any other attracting cycles will be of the form $[0,a)$ or $(b,1]$, so will contain 0 or 1. □

Example 7.2.2 We now see that the logistic map $L_\mu(x) = \mu x(1-x)$, $0 < \mu < 4$, $x \in [0,1]$, has at most one attracting periodic cycle. If $0 < \mu \leq 1$, 0 is the only attracting fixed point, having basin of attractions $[0,1]$. For $1 < \mu < 4$, L_μ has exactly one critical point $x_0 = 1/2$.

Since $L'_\mu(0) = \mu > 1$, the fixed point 0 is unstable; therefore $[0,a)$ cannot be a basin of attraction. Furthermore, $L_\mu(1) = 0$ and hence $(b,1]$ is not a basin of attraction either. Since $SL_\mu(x) < 0$ everywhere (at $x = 1/2$, $\lim_{x\to 1/2} L_\mu(x) = -\infty$), we conclude that there is at most one attracting periodic cycle in $(0,1)$ and the result follows.

If we look at the bifurcation diagram of the logistic map, it follows, for example that where we see six horizontal lines, we have an attracting 6-cycle, and not two attracting 3-cycles.

Remark 7.2.3 In a similar way we get Singers Theorem [45], proved by David Singer in 1978. This theorem is actually a real version of a theorem about complex polynomials proved by the French mathematician Gaston Julia in 1918 [22]:

نظرية سنجر

Suppose that f is a ① C^3 map on a ② closed interval I such that ③ $Sf(x) < 0$, for all $x \in I$. If f has n critical points in I, then f has at most $n+2$ attracting cycles.

p-70 S.E.

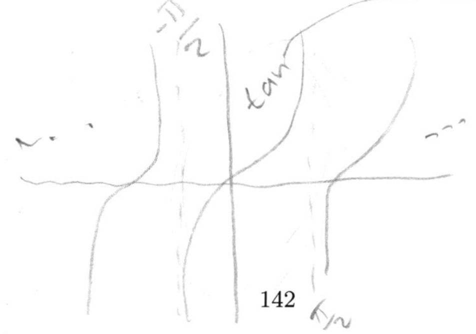
اختبار لإيجاد حد أعلى لعدد النقاط الدورية الجاذبة.

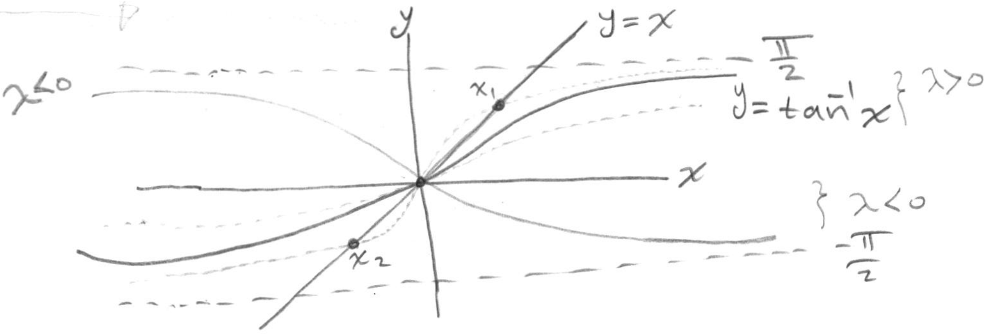
Remark.

نحصل عليه من خلال الإثبات p.70 S.E.

If $Sf < 0$ and the immediate basin of attraction $J = (c,d)$ of periodic point p is bounded, then p must attract a critical point.

Example. $f(x) = 1 - 2x^2$ on $[-1,1]$ has no attracting periodic pts.





عدد نقاط حرجة $n=0$ Since $G'(x) \neq 0$ because $\lambda \neq 0$ $G'(x) = \frac{\lambda}{1+x^2} \neq 0$

Example 7.2.4 Let $G(x) = \lambda \arctan(x)$, $\lambda \neq 0$. Then $G'(x) = \lambda/(1+x^2)$. Clearly $G(x)$ has no critical points. Now, if $|\lambda| < 1$, then $c = 0$ is an asymptotically stable fixed point with basin of attraction $(-\infty, \infty)$. If $\lambda > 1$, then G has two attracting fixed points x_1 and x_2 with basins of attraction $(-\infty, 0)$ and $(0, \infty)$, respectively. Finally, if $\lambda < -1$, then G has an attracting 2-cycle $\{\bar{x}_1, \bar{x}_2\}$ with basin of attraction $(-\infty, 0) \cup (0, \infty)$. $= \mathbb{R} - \{0\}$

Exercises 7.2

1. If $G(x) = \lambda \arctan(x)$, $(\lambda \neq 0)$, show that the Schwarzian derivative is

$$SG(x) = \frac{-2}{(1+x^2)^2}.$$

Use this to verify the conclusion from Example 7.2.4.

$2(2x)(1+x^2)$

$$G'(x) = \frac{\lambda}{1+x^2}, \quad G''(x) = \frac{-\lambda(2x)}{(1+x^2)^2}, \quad G'''(x) = \frac{(1+x^2)^2(-2\lambda) + \lambda(2x)4}{(1+x^2)^4}$$

$$SG(x) = \frac{G'''(x)}{G'(x)} - \frac{3}{2}\left[\frac{G''(x)}{G'(x)}\right]^2$$

$$= \frac{\dfrac{-2\cancel{\lambda}(1+x^2)^{\cancel{2}} + 8\cancel{\lambda} x^2 \cancel{(1+x^2)}}{(1+x^2)^{\cancel{4}\,3\,2}}}{\dfrac{\cancel{\lambda}}{\cancel{1+x^2}}} - \frac{3}{2}\left[\frac{\dfrac{-2\cancel{\lambda} x}{(1+x^2)^{\cancel{2}}}}{\dfrac{\cancel{\lambda}}{\cancel{1+x^2}}}\right]^2$$

$$= \frac{-2(1+x^2) + 8x^2 - \frac{3}{\cancel{2}}(\cancel{4}^{2}x^2)}{(1+x^2)^2}$$

$$= \frac{-2 - 2x^2 + 2x^2}{(1+x^2)^2} = \frac{-2}{(1+x^2)^2} < 0 \quad \forall x \in \mathbb{R}$$

and $G \in C^3$ $\forall x \in \mathbb{R}$ so $G \in C^3$ on any closed interval. $SG < 0$ on any closed interval, $n = 0 =$ the number of critical points, So by (Singers thm.) G has at most two attracting cycles.

السنة الرابعة رياضيات

ch.
4

Dynamical Systems نظم ديناميكية

Stability of Two-Dimensional Maps

من كتاب [Discrete Chaos (2^{nd} ed.) with applications in Science and Engineering]
Saber N. Elaydi
Chapman & Hall/CRC 2008.

Is evolution a matter of survival of the fittest or survival of the most stable?

A. M. Waldrop

تذكير بتعاريف القيم الذاتية والمتجهات الذاتية (القيم الذاتية هي جذور المعادلة المميزة)

وخاصة لـ $A_{2\times2}$ $|A-\lambda I|=0$. والمتجه الذاتي V متجه غير صفري يحقق $AV=\lambda V$ و $V\neq 0$

4.1 Linear Maps vs. Linear Systems

$a_{11}+a_{22}=tr(A)=\lambda_1+\lambda_2$
$\det(A)=a_{11}a_{22}-a_{12}a_{21}$

Recall from linear algebra that a map $L:\mathbb{R}^2\to\mathbb{R}^2$ is called a linear transformation if

1. $L(U_1+U_2)=L(U_1)+L(U_2)$ for $U_1,U_2\in\mathbb{R}^2$
2. $L(\alpha U)=\alpha L(U)$ for $U\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$.

Moreover, it is always possible to represent ~~f~~ L (with a given basis for $\mathbb{R}^2$) by a matrix A. A typical example is

$$L\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}$$

which may be written in the form

$$L\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

or

$$L(U)=AU, \tag{4.1}$$

where $U=\begin{pmatrix}x\\y\end{pmatrix}$ and $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$. $\Rightarrow |A-\lambda I|=\begin{vmatrix}a-\lambda&b\\c&d-\lambda\end{vmatrix}=0$ $\Rightarrow$

By iterating L, we conclude that $L^n(U)=A^nU$. Hence, the orbit of U under ~~f~~ L is given by

$$\{U, AU, A^2U,\ldots,A^nU,\ldots\} \tag{4.2}$$

Thus, to compute the orbit of U, it suffices to compute A^nU for $n\in\mathbb{Z}^+$.

$$(a-\lambda)(d-\lambda) - bc = 0$$

$\therefore \lambda^2 - (a+d)\lambda + ad - bc = 0$ المعادلة المميزة

So $\lambda_{1,2} = \frac{1}{2}\left[a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right]$

Another way of looking at the same problem is by considering the following two-dimensional system of difference equations

$$\begin{aligned} x(n+1) &= ax(n) + by(n) \\ y(n+1) &= cx(n) + dy(n), \end{aligned} \tag{4.3}$$

or

$$U(n+1) = AU(n). \tag{4.4}$$

By iteration, one may show that the solution of Equation (4.4) is given by

$$U(n) = A^n U(0). \tag{4.5}$$

So, if we let $U_0 = U(0)$, then $L^n(U_0) = U(n)$.

The form of Equation (4.3) is more convenient when we are considering applications in biology, engineering, economics, and so forth. For example, $x(n)$ and $y(n)$ may represent the population sizes at time period n of two competitive cooperative species, or preys and predators. الفريسة و المفترس

In the next section, we will develop the necessary machinery to compute A^n for any matrix of order two. The general theory may be found in [32, 33, 60].

4.2 Computing A^n

Consider a matrix $A = (a_{ij})$ of order 2×2. Then, $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of A and its zeros are called the **eigenvalues** of A. Associated with each eigenvalue λ of A a nonzero eigenvector $V \in \mathbb{R}^2$ with $AV = \lambda V$.

Example 4.1

Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}. \quad \square$$

SOLUTION First we find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ or

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

which is

$$\lambda^2 - 6\lambda + 5 = 0.$$

Hence, $\lambda_1 = 1$ and $\lambda_2 = 5$. To find the corresponding eigenvector V_1, we solve the vector equation $AV_1 = \lambda V_1$ or $(A - \lambda_1 I)V_1 = 0$.

For $\lambda_1 = 1$, we have

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $v_{11} + 3v_{21} = 0$. Thus, $v_{11} = -3v_{21}$. So, if we let $v_{21} = 1$, then $v_{11} = -3$. It follows that the eigenvector V_1 corresponding to λ_1 is given by $V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 5$, the corresponding eigenvector may be found by solving the equation $(A - \lambda_2 I)V_2 = 0$. This yields

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $-3v_{12} + 3v_{22} = 0$ or $v_{12} = v_{22}$. It is then appropriate to let $v_{12} = v_{22} = 1$ and hence $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. ∎

To find the general form for A^n for a general matrix A is a formidable task even for a 2×2 matrix such as in Example 4.1. Fortunately, however, we may be able to transform a matrix A to another simpler matrix B whose nth power B^n can easily be computed. The essence of this process is captured in the following definition.

DEFINITION 4.1 *The matrices A and B are said to be similar if there exists a nonsingular[1] matrix P such that*

$$\boxed{A = PBP^{-1}} \equiv P^{-1}AP = B. \equiv \overset{①}{A}P = P\overset{②}{B}$$

تكافؤ

We note here that the relation "similarity" between matrices is an equivalence relation, i.e.,

1. A is similar to A. $A \underset{I}{\approx} A$ (reflexivity) انعكاسية
2. If A is similar to B then B is similar to A. $A \approx B \Rightarrow B \approx A$ (Symmetry) تناظرية
3. If A is similar to B and B is similar to C, then A is similar to C. متعدية transiti
   $A \approx B$ and $B \approx C \Longrightarrow A \approx C$ (transitivity)

The most important feature of similar matrices, however, is that they possess the same eigenvalues.

(invertible)

[1] A matrix P is said to be nonsingular if its inverse P^{-1} exists. This is equivalent to saying that det $P \neq 0$, where det denotes determinant.

THEOREM 4.1
Let A and B be two similar matrices. Then A and B have the same eigenvalues.

PROOF Suppose that $P^{-1}AP = B$ or $A = PBP^{-1}$. Let λ be an eigenvalue of A and V be the corresponding eigenvector. Then, $\lambda V = AV = PBP^{-1}V$. Hence, $B(P^{-1}V) = \lambda(P^{-1}V)$. Consequently, λ is an eigenvalue of B with $P^{-1}V$ as the corresponding eigenvector. ∎

The notion of similarity between matrices corresponds to linear conjugacy, which we have encountered in Chapter 3. In other words, two linear maps are conjugate if their corresponding matrix representations are similar. Thus, the linear maps L_1, L_2 on $\mathbb{R}^2$ are linearly conjugate if there exists an invertible map h such that

$$L_1 \circ h = h \circ L_2$$

or

$$h^{-1} \circ L_1 \circ h = L_2.$$

The next theorem tells us that there are three simple "canonical" forms for 2×2 matrices.

THEOREM 4.2
Let A be a 2×2 real matrix. Then A is similar to one of the following matrices:

1. $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\lambda_1 \neq \lambda_2$ are real
2. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $\lambda_1 = \lambda_2 = \lambda$
3. $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $\lambda_{1,2} = \alpha \pm i\beta$

PROOF Suppose that the eigenvalues λ_1 and λ_2 are real. Then, we have two cases to consider. The first case is where $\lambda_1 \neq \lambda_2$. In this case, we may easily show that the corresponding eigenvectors V_1 and V_2 are linearly independent (Problem 10). Hence, the matrix $P = (V_1, V_2)$, i.e., the matrix P whose columns are these eigenvectors, is nonsingular. Let $P^{-1}AP = J = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then,

الاطلاع وأخذ الصيغ

$$AP = PJ. \tag{4.6}$$

Comparing both sides of Equation (4.6), we obtain

$$AV_1 = eV_1 + gV_2.$$

Hence,

$$\lambda_1 V_1 = eV_1 + gV_2.$$

Thus, $e = \lambda_1$ and $g = 0$.

Similarly, one may show that $f = 0$ and $h = \lambda_2$. Consequently, J is a diagonal matrix of the form (a).

The second case is where $\lambda_1 = \lambda_2 = \lambda$. There are two subcases to consider here. The first subcase occurs if we are able to find two linearly independent eigenvectors V_1 and V_2 corresponding to the eigenvalue λ. This subcase is then reduced to the preceding case. We note here that this scenario happens when $(A - \lambda I)V = 0$ for all $V \in \mathbb{R}^2$. In particular, one may let $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which are clearly linearly independent.

The second subcase occurs when there exists a nonzero vector $V_2 \in \mathbb{R}^2$ such that $(A - \lambda I)V_2 \neq 0$. Equivalently, we are able to find only one eigenvector (not counting multiples) V_1 with $(A - \lambda I)V_1 = 0$. In practice, we find V_2 by solving the equation

$$(A - \lambda I)V_2 = V_1.$$

The vector V_2 is called a generalized eigenvector of A. Note that $AV_1 = \lambda V_1$ and $AV_2 = \lambda V_2 + V_1$. Now, we let $\boxed{P = (V_1, V_2)}$ and $P^{-1}AP = J$. Then,

فقط لما $\lambda_1=\lambda_2$ $\boxed{(A-\lambda I)V_2 = V_1}$

$$AP = PJ. \tag{4.7}$$

لايجاد P

Comparing both sides of Equation (4.7) yields

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{4.8}$$

The matrix J is in a **Jordan form**.

Next, we assume that A has a complex eigenvalue $\lambda_1 = \alpha + i\beta$. Since A is assumed to be real, it follows that the second eigenvalue λ_2 is a conjugate of λ_1, that is, $\lambda_2 = \alpha - i\beta$. Let $V = V_1 + iV_2$ be the eigenvector corresponding to λ_1. Then,

$$\begin{aligned} AV &= \lambda_1 V \\ A(V_1 + iV_2) &= (\alpha + i\beta)(V_1 + iV_2). \end{aligned}$$

Hence,

$$\begin{aligned} AV_1 &= \alpha V_1 \quad \beta V_2 \\ AV_2 &= \beta V_1 + \alpha V_2, \end{aligned}$$

$\boxed{A = PJP^{-1}}$

letting $P = (V_1, V_2)$ we get $P^{-1}AP = J$. Hence,

$$AP = PJ. \tag{4.9}$$

Comparison of both sides of Equation (4.9) yields

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{4.10}$$

Theorem 4.2 gives us a simple method of computing the general form of A^n for any 2×2 real matrix. In the first case, when $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, we have

$$\begin{aligned} A^n &= (PDP^{-1})^n \\ &= PD^nP^{-1} \\ &\boxed{A^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}.} \end{aligned} \tag{4.11}$$

In the second case, when $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then

$$\begin{aligned} A^n &= PJ^nP^{-1} \\ &\boxed{A^n = P \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} P^{-1}.} \end{aligned} \tag{4.12}$$

Equation (4.12) may be easily proved by mathematical induction (Problem 11).

In the third case, we have $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Let $\omega = \arctan(\beta/\alpha)$. Then $\cos\omega = \alpha/|\lambda_1|$, $\sin\omega = \beta/|\lambda_1|$. Now, we write the matrix J in the form

$$J = |\lambda_1| \begin{pmatrix} \alpha/|\lambda_1| & \beta/|\lambda_1| \\ -\beta/|\lambda_1| & \alpha/|\lambda_1| \end{pmatrix} = |\lambda_1| \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

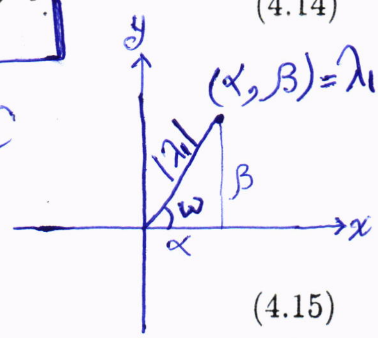
By mathematical induction one may show that (Problem 11)

$$J^n = |\lambda_1|^n \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}. \tag{4.13}$$

and thus

$$\boxed{A^n = |\lambda_1|^n P \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix} P^{-1}.} \tag{4.14}$$

■



C

Example 4.2
Solve the system of difference equations

$$X(n+1) = AX(n) \tag{4.15}$$

$X_{n+1} = A^2 X_{n-1}$
$= A^3 X_{n-2}$
$= \cdots$
$= A^{n+1} X_0$

where

$$A = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ □

وكان السؤال عن
الصيغة العامة لمسار X_0
$O(X_0)$

SOLUTION The eigenvalues of A are repeated: $\lambda_1 = \lambda_2 = 2$. The only eigenvector that we are able to find is $V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. To construct P we need to find a generalized eigenvector V_2. This is accomplished by solving the equation $(A-2I)V_2 = V_1$. Then, V_2 may be taken as any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, with $3y - 2x = 1$. We take $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now if we put $P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, then $P^{-1}AP = J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Thus, the solution of Equation (4.15) is given by

$\boxed{A^n = PJ^nP^{-1}}$

$$\begin{aligned} X(n) &= PJ^nP^{-1}x(0) \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 2^n \begin{pmatrix} 1-3n \\ -2n \end{pmatrix}. \end{aligned}$$ ∎

REMARK 4.1 If a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(X_0) = AX_0$, then $f^n(X_0) = A^nX_0 = PJ^nP^{-1}X_0$. In particular, if $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $f^n(X_0) = 2^n \begin{pmatrix} 1-3n \\ -2n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$. ∎

الصيغة العامة لعناصر
في مسار X_0 $O(X_0)$

Exercises - (4.1 and 4.2)

In Problems 1–5, find the eigenvalues and eigenvectors of the matrix A and compute A^n.

1. $A = \begin{pmatrix} -4.5 & 5 \\ -7.5 & 8 \end{pmatrix}$
2. $A = \begin{pmatrix} 4.5 & -1 \\ 2.25 & 1.5 \end{pmatrix}$
3. $A = \begin{pmatrix} 8/3 & 1/3 \\ -4/3 & 4/3 \end{pmatrix}$
4. $A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$
5. $A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$

Remarks.
① If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $ad-bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

② If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $T = tr = tr(A) = a+d =$ trace أثر A
$D = det(A) = |A| = ad-bc$. المحدد
and $tr(A) = \lambda_1 + \lambda_2$, $det(A) = \lambda_1\lambda_2$

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(X) = AX$, where A is as in Problem 1. Find $L^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

7. Solve the difference equation $X(n+1) = AX(n)$, where A is as in Problem 3, and $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

8. Solve the difference equation $X(n+1) = AX(n)$, where A is as in Problem 4, and $X(0) = X_0$.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(X) = AX$, with A as in Problem 5. Find $f^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

10. Let A be a 2×2 matrix with distinct real eigenvalues. Show that the corresponding eigenvectors of A are linearly independent.

11. (a) If $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, show that $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

    (b) If $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, show that $J^n = |\lambda|^n \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}$, where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$, $\omega = \arctan\left(\frac{\beta}{\alpha}\right)$.

12. Let a matrix A be in the form

$$A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}.$$

    (a) Show that if A has distinct eigenvalues λ_1 and λ_2, then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

    where $P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$.

    (b) Show that if A has a repeated eigenvalue λ, then

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

    where $P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$.

    (c) Show that if A has complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then

$$P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

    where $P = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$.

4.3 Fundamental Set of Solutions

Consider the linear system

$$X(n+1) = AX(n), \tag{4.16}$$

where A is a 2×2 matrix. Then, two solutions $X_1(n)$ and $X_2(n)$ of Equation (4.16) are said to be linearly independent if $X_2(n)$ is not a scaler multiple of $X_1(n)$ for all $n \in \mathbb{Z}^+$. In other words, if $c_1X_1(n)+c_2X_2(n) = 0$ for all $n \in \mathbb{Z}^+$, then $c_1 = c_2 = 0$. A set of two linearly independent solutions $\{X_1(n), X_2(n)\}$ is called a fundamental set of solutions of Equation (4.16).

DEFINITION 4.2 *Let $\{X_1(n), X_2(n)\}$ be a fundamental set of solutions of Equation (4.16). Then*

$$X(n) = k_1X_1(n) + k_2X_2(n), \quad k_1, k_2 \in \mathbb{R} \tag{4.17}$$

(linear Cobination)

is called a general solution of Equation (4.16).

الحل العام هو تركيب خطي من حلين مستقلين خطيًا.

Finding $X_1(n)$ and $X_2(n)$ is generally an easy task. We now give an explicit derivation.

In the sequel λ_1, λ_2 denote the eigenvalues of A; V_1, V_2 are the corresponding eigenvectors of A.

We have three cases to consider.

لكل حالة قانونها

Case (i)

Suppose that $P^{-1}AP = J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then a general solution may be given by

$$\begin{aligned} X(n) &= A^nX(0) = PJ^nP^{-1}X(0) \\ &= (V_1,\ V_2)\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

where $\boxed{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = P^{-1}X(0)}$. Then,

$$\boxed{X(n) = k_1\lambda_1^nV_1 + k_2\lambda_2^nV_2.} \tag{4.18}$$

Here, $X_1(n) = \lambda_1^nV_1$ and $X_2(n) = \lambda_2^nV_2$ constitute a fundamental set of solutions since in this case V_1 and V_2 are linearly independent eigenvectors. Note that one may check directly that $\lambda_1^nV_1$ and $\lambda_2^nV_2$ are indeed solutions of Equation (4.16) (Problem 13a).

Case (ii)

Suppose that $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Then, a general solution may be given by

$$X(n) = PJ^nP^{-1}X(0)$$

$$= (V_1,\ V_2)\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = P^{-1}X(0)$$

$$\boxed{X(n) = k_1\lambda^n V_1 + k_2(n\lambda^{n-1}V_1 + \lambda^n V_2)} \tag{4.19}$$

Hence, $X_1(n) = \lambda^n V_1$ and $X_2(n) = \lambda^n V_2 + n\lambda^{n-1}V_1$ constitute a fundamental set of solutions of Equation (4.16) (Problem 13b).

Case (iii)

Suppose that $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. If $\omega = \arctan(\beta/\alpha)$, then the general solution may be given by

$$X(n) = PJ^nP^{-1}X(0)$$

$$= \underbrace{(V_1\, V_2)}_{P}\underbrace{|\lambda_1|^n\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}}_{J^n}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = P^{-1}X(0)$$

$$\boxed{\begin{aligned} X(n) = |\lambda_1|^n[(k_1\cos n\omega + k_2\sin n\omega)V_1 \\ +(-k_1\sin n\omega + k_2\cos n\omega)V_2]. \end{aligned}} \tag{4.20}$$

Hence, $X_1(n) = |\lambda_1|^n[(\cancel{k_1}\cos n\omega)V_1 - (\cancel{k_1}\sin(n\omega))V_2]$ and $X_2(n) = (|\lambda_1|^n[(\cancel{k_2}\sin(n\omega))V_1 + (\cancel{k_2}\cos(n\omega)]V_2$ constitute a fundamental set of solutions (Problem 13c).

where $\boxed{V_1 = Re(V)}$ $\boxed{V_2 = Im(V)}$

and $\boxed{X(0) = k_1V_1 + k_2V_2}$

Example 4.3
Solve the system of difference equations

$$X(n+1) = AX(n),\ X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}. \quad \square$$

SOLUTION The eigenvalues of A are $\lambda_1 = -2+3i$ and $\lambda_2 = -2-3i$. The corresponding eigenvectors are $V = \begin{pmatrix} -1 \\ i \end{pmatrix}$ and $\overline{V} = \begin{pmatrix} -1 \\ -i \end{pmatrix}$, respectively.

This time, we take a short cut and use Equation (4.20). The vectors V_1 and V_2 referred to in this formula are the real part of V, $V_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and

the imaginary part of V, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, $|\lambda_1| = \sqrt{13}$, $\omega = \arctan(\frac{-3}{2}) \approx 123.69°$. Thus,

$$X(n) = (13)^{n/2}\Big[(k_1 \cos n\omega + k_2 \sin n\omega)\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$+ (-k_1 \sin n\omega + k_2 \cos n\omega)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]$$

$$X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, $k_1 = \overset{-1}{1}, k_2 = 2$. Thus,

$$X(n) = (13)^{n/2}\Big[\overset{-1}{(}\cos n\omega + 2 \sin n\omega)\begin{pmatrix} \overset{-1}{1} \\ 0 \end{pmatrix} + (\overset{+}{-} \sin n\omega + 2 \cos n\omega)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]$$

$$= (13)^{n/2}\begin{pmatrix} \overset{+}{-} \cos n\omega - 2 \sin n\omega \\ \underset{+}{-} \sin n\omega + 2 \cos n\omega \end{pmatrix}.$$

4.4 Second-Order Difference Equations

A second-order difference equation with constant coefficients is a scalar equation of the form

$$u(n+2) + p_1 u(n+1) + p_2 u(n) = 0 \tag{4.21}$$

Although one may solve this equation directly, it is sometimes beneficial to convert it to a two-dimensional system. The trick is to let $u(n) = x_1(n)$ and $u(n+1) = x_2(n)$.

Then we have

$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = -p_2 x_1(n) - p_1 x_2(n)$$

which is of the form

$$X(n+1) = AX(n) \tag{4.22}$$

where

$$X(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}.$$

The characteristic equation of A is given by

$$\lambda^2 + p_1\lambda + p_2 = 0. \tag{4.23}$$

Observe that we may obtain the characteristic Equation (4.23) by letting $u(n) = \lambda^n$ in Equation (4.21). Thus, if λ_1 and λ_2 are the roots of Equation (4.23), then $u_1(n) = \lambda_1^n$ and $u_2(n) = \lambda_2^n$ are solutions of Equation (4.21).

Using Eqs. (4.18), (4.19), and (4.20), we can make the following conclusions:

1. If $\lambda_1 \neq \lambda_2$ and both are real, then the general solution of Equation (4.21) is given by
$$\boxed{u(n) = c_1\lambda_1^n + c_2\lambda_2^n,} \tag{4.24}$$

2. If $\lambda_1 = \lambda_2 = \lambda$, then the general solution of Equation (4.21) is given by
$$\boxed{u(n) = c_1\lambda^n + c_2 n\lambda^n,} \tag{4.25}$$

3. If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, then the general solution of Equation (4.21) is given by
$$\boxed{u(n) = |\lambda_1|^n(c_1 \cos n\omega + c_2 \sin n\omega),} \tag{4.26}$$
where $\omega = \arctan(\beta/\alpha)$.

for Second order d.e.

Example 4.4
Solve the second-order difference equation
$$x(n+2) + 6x(n+1) + 9x(n) = 0, \;\; x(0) = 1, \;\; x(1) = 0. \qquad \square$$

SOLUTION The characteristic equation associated with the equation is given by $\lambda^2 + 6\lambda + 9 = 0$.

Hence, the characteristic roots are $\lambda_1 = \lambda_2 = -3$. The general solution is given by
$$\begin{aligned} x(n) &= \overset{c_1}{\cancel{9}}(-3)^n + c_2 n(-3)^n \\ x(0) &= 1 = c_1 \\ x(1) &= 0 = -3c_1 - 3c_2. \end{aligned}$$

Thus, $c_2 = -1$ and, consequently,
$$\begin{aligned} x(n) &= (-3)^n - n(-3)^n \\ &= (-3)^n(1-n) \qquad \blacksquare \end{aligned}$$

Exercises - (4.3 and 4.4)

1. Solve the system
$$\begin{aligned} x_1(n+1) &= -x_1(n) + x_2(n) \\ x_2(n+1) &= 2x_2(n) \\ \text{with } x_1(0) = 1, &\;\; x_2(0) = 2. \end{aligned}$$

2. Find the general solution of the system

$$X(n+1) = AX(n), \quad \text{where} \quad A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}.$$

3. Solve the problem $X(n+1) = AX(n)$, where $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

4. Solve the system

$$X(n+1) = AX(n), \quad \text{with } A = \begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

5. Solve the system $x(n+1) = Ax(n)$, with $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

6. Solve the difference equation

$$x(n+2) - 5x(n+1)_6x(n) = 0, \; x(0) = 2$$

   (a) By converting it to a system,

   (b) Directly as it is.

7. Solve the difference equation

$$F(n+2) = F(n+1) + F(n), \; F(1) = 1, \; F(2) = 1.$$

   (This is called the Fibonacci sequence.)

8. The Chebyshev polynomials of the first and second kind are defined as follows:

$$\begin{aligned} T_n(x) &= \cos(n\cos^{-1}(x)), \\ U_n(x) &= \frac{1}{\sqrt{1-x^2}} \sin[(n+1)\cos^{-1}(x)], \quad \text{for } |x| < 1. \end{aligned}$$

   (a) Show that $T_n(x)$ satisfies the difference equation

$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0, \; T_0(x) = 1, \; T_1(x) = x.$$

   (b) Solve for $T_n(x)$.

   (c) Show that

$$U_{n+2}(x) - 2xU_{n+1}(x) + U_n(x) = 0, \; U_0(x) = 1, \; U_1(x) = 2x.$$

   (d) Write down the first four terms of $T_n(x)$ and $U_n(x)$.

9. Solve the equation $x(n+2) + 16x(n) = 0$.

10. Let A be a 2×2 real matrix with distinct eigenvalues λ_1 and λ_2. Prove that the corresponding eigenvectors V_1 and V_2 are linearly independent.

11. Let A be a 2×2 real matrix with a repeated eigenvalue λ. Let V_1 be an eigenvector corresponding to λ and let V_2 be a generalized eigenvector. Show that V_1 and V_2 are linearly independent.

12. Let A be a 2×2 real matrix with complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. Suppose that $V = V_1 + iV_2$ is the eigenvector corresponding to λ_1. Prove that the matrix $P = (V_1, V_2)$ is nonsingular. (*Hint: It suffices to show that V_1 and V_2 are linearly independent.*)

13. (a) Show that $X_1(n)$ and $X_2(n)$, obtained from Equation (4.18), are solutions of Equation (4.16).
    (b) Show that $X_1(n)$ and $X_2(n)$, obtained from Equation (4.19), are solutions of Equation (4.16).
    (c) Show that $X_1(n)$ and $X_2(n)$, obtained from Equation (4.20), are solutions of Equation (4.16).

In Problems 14 and 15, consider the nonhomogeneous equation

$$Y(n+1) = AY(n) + g(n) \tag{4.27}$$

where A is a 2×2 matrix and g is a function defined on $\mathbb{Z}^+$.

14. Show that

$$Y(n) = A^n Y(0) + \sum_{k=0}^{n-1} A^{n-k-1} g(k). \tag{4.28}$$

(This is called the variation of constants formula.)

15. Use Formula (4.28) to find the solution of Equation (4.27) with

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad g(n) = \begin{pmatrix} n \\ 1 \end{pmatrix}, \quad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

16. Solve the equation $y(n+2) - 5y(n+1) + 4y(n) = 4^n$.

Poincaré

4.5 Phase Space Diagrams

فضاء الطور portrait الطور (صورة، طور)

One of the best graphical methods to illustrate the various notions of stability is the phase portrait or the phase space diagram. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a given map. Then, starting from an initial point $X_0 = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$, we plot the

sequence of point $X_0, f(X_0), f^2(X_0), f^3(X_0), \ldots$ and then connect the points by straight lines. An arrow is placed on these connecting lines to indicate the direction of the motion on the orbit. In many instances, we need to be prudent in choosing our initial points in order to get a better phase portrait.

In this section, we consider linear systems for which $f(X) = AX$, where A is a 2×2 matrix. Observe that if $\boxed{A - I}$ is nonsingular, i.e., $\det(A - I) \neq 0$, then the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only fixed point of the map f. Equivently, $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only fixed point of the system

$$X(n+1) = AX(n) \tag{4.29}$$

مفروض

As stipulated in Theorem 4.2, there exists a nonsingular matrix P such that $P^{-1}AP = J$ where J is one of the forms (1), (2), or (3) in Theorem 4.2. If we let

$$X(n) = \overset{-1}{P}Y(n) \tag{4.30}$$

in Equation (4.29), we obtain

$X(n+1) = P J P^{-1} X(n)$

$P^{-1} X(n+1) = J P^{-1} X(n)$

$$Y(n+1) = JY(n). \tag{4.31}$$

Our plan here is to draw the phase portrait of Equation (4.31), then use the transformation (4.30) to obtain the phase portrait of the original system (4.29).

(I). We begin our discussion by assuming that J is in the diagonal form $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1 and λ_2 are not necessarily distinct. Here we have two linearly independent solutions:

$$Y_1(n) = \lambda_1^n V_1, \quad \text{and} \quad Y_2(n) = \lambda_2^n V_2, \quad \text{where} \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are the eigenvectors of A corresponding to the eigenvalues λ_1 and λ_2, respectively.

Observe that $Y_1(n)$ is a multiple of V_1, and thus must stay on the line emanating from the origin in the direction of V_1; in this case, the x axis. Similarly, $Y_2(n)$ must stay on the line passing through the origin and in the direction of V_2; in this case, the y axis. These two solutions are called straight line solutions. The general solution is given by

$$Y(n) = k_1 \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Y(0) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \tag{4.32}$$

We have the following cases to consider:

1. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then all solutions tend to the origin as $n \to \infty$. Observe that if $|\lambda_1| < |\lambda_2| < 1$, then, $|\lambda_1^n|$ goes to zero faster than $|\lambda_2^n|$.

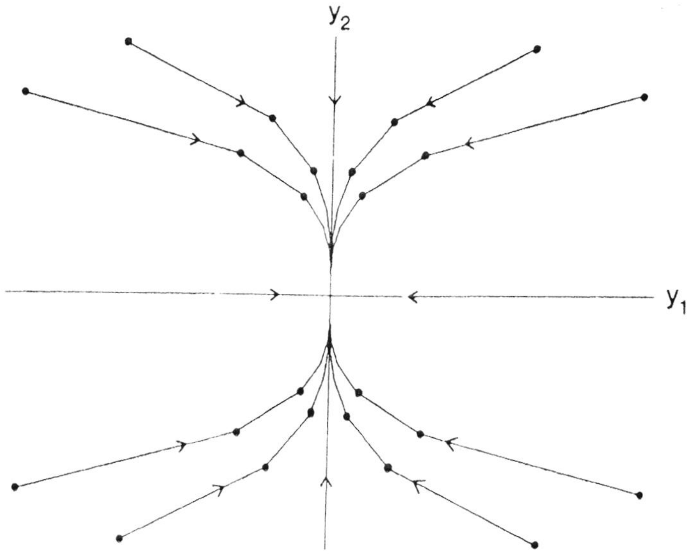


FIGURE 4.1a
(a) A sink: $0 < \lambda_1 < \lambda_2$.

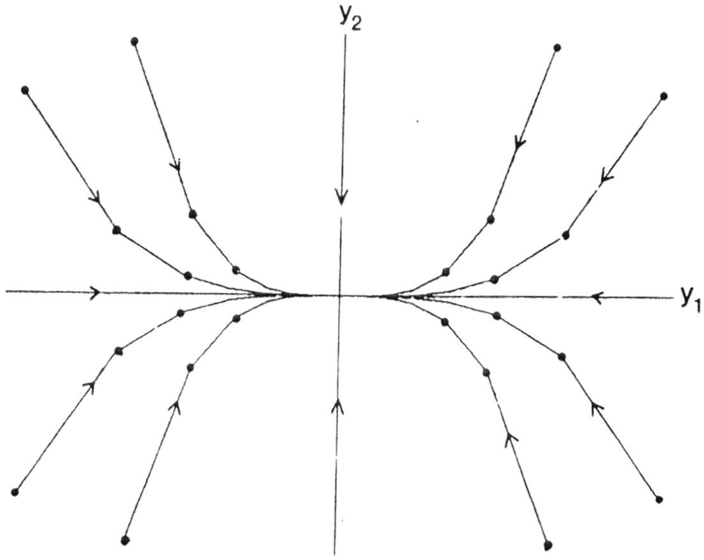


FIGURE 4.1b
(b) A sink: $0 < \lambda_2 < \lambda_1 < 1$.

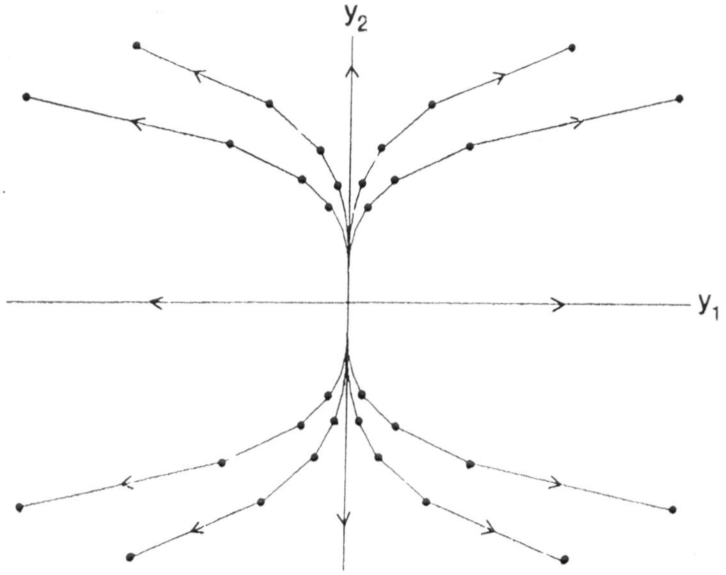


FIGURE 4.2a
(a) A source: $\lambda_1 > \lambda_2 > 1$.

And consequently, any solution $Y(n)$ in the form (4.31) is asymptotic to the straight line solution $Y_2(n) = \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (see Fig. 4.1a).

On the other hand, if $|\lambda_1| > |\lambda_2|$, then $Y(n)$ is asymptotic to $Y_1(n) = \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (see Fig. 4.1b).

Phase portraits 4.1a and 4.1b are called <u>sinks</u>.

2. If $|\lambda_1| > 1$, and $|\lambda_2| > 1$, then we obtain an source as illustrated in Figs. 4.2a and 4.2b.

   Note that if $|\lambda_1| > |\lambda_2| > 1$, then $Y(n)$ is asymptotic to $Y_2(n) = \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (the y axis) when $n \to -\infty$ and is dominated by $Y_1(n) = \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ when $n \to \infty$.

3. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then we obtain a saddle (Fig. 4.3). In this case, when $n \to \infty$ $Y(n)$, is asymptotic to $Y_2(n) = \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $n \to \infty$ and is asymptotic to $Y_1(n) = \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $n \to -\infty$. Similar analysis is readily available for the case $|\lambda_1| > 1$ and $|\lambda_2| < 1$.

4. If $\lambda_1 = \lambda_2$, then

$$Y(n) = k_1\lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2\lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda^n \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

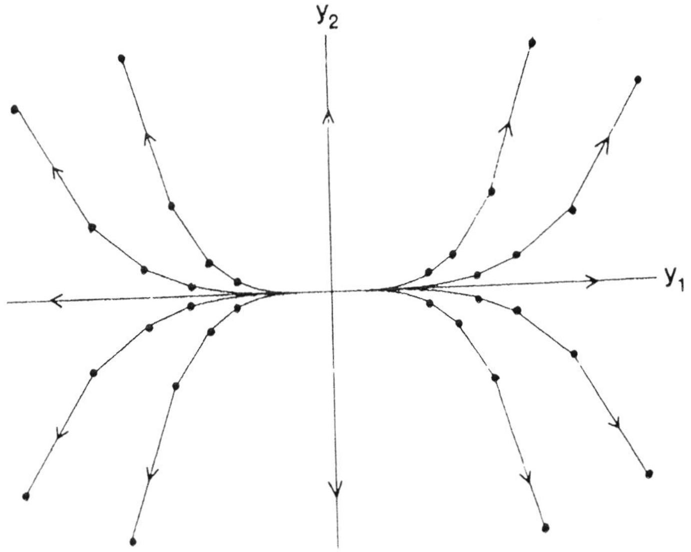


FIGURE 4.2b
(b) A source: $\lambda_2 > \lambda_1 > 1$.

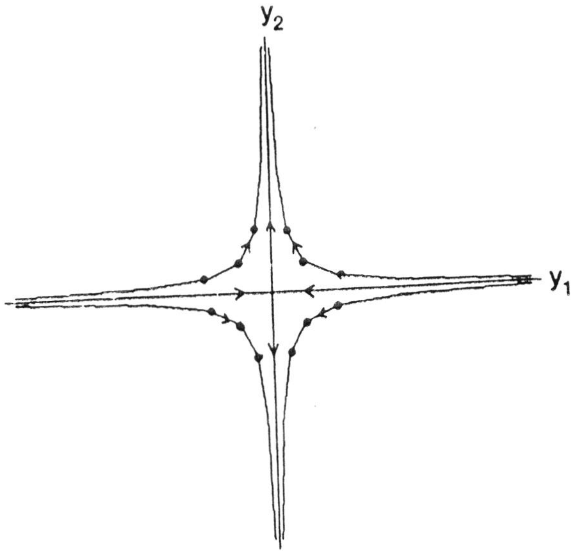


FIGURE 4.3
A saddle: $0 < \lambda_1 < 1, \lambda_2 > 1$.

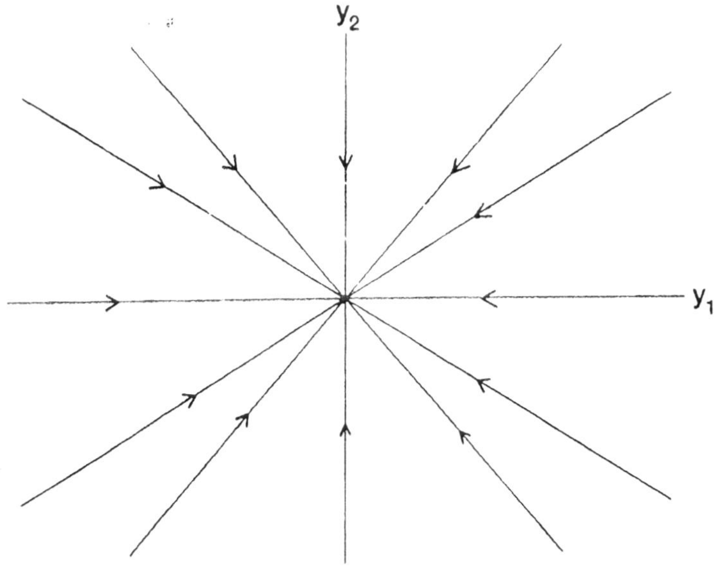


FIGURE 4.4
A sink: $0 < \lambda_1 < \lambda_2 < 1$.

Hence, every solution $Y(n)$ lies on a line passing through the origin with a slope k_2/k_1 (see Fig. 4.4).

Observe that in each of the four subcases, the presence of a negative eigenvalue will cause the solution $Y(n)$ to oscillate around the origin and the phase portrait will not look as nice as in Figs. 4.1a–4.4.

(II). Suppose that J is in the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Then, we only have one straight line solution, $Y_1(n) = \lambda^n V_1 = \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The general solution is given by

$$\begin{aligned} Y(n) &= k_1 \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \left(n\lambda^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= (k_1\lambda + k_2 n)\lambda^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Now, if $|\lambda| < 1$, then, $Y(n) \to 0$ as $n \to \infty$, since $\lim_{n\to\infty} n\lambda^{n-1} = 0$ (by L'Hopital Rule). Since the term $k_1 \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tends to the origin, as $n \to \infty$, faster than the term $(k_1\lambda + k_2 n)\lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, our solution $Y(n)$ tends to the origin asymptotic to the x axis (see Fig. 4.5a). In this case, the origin is called a

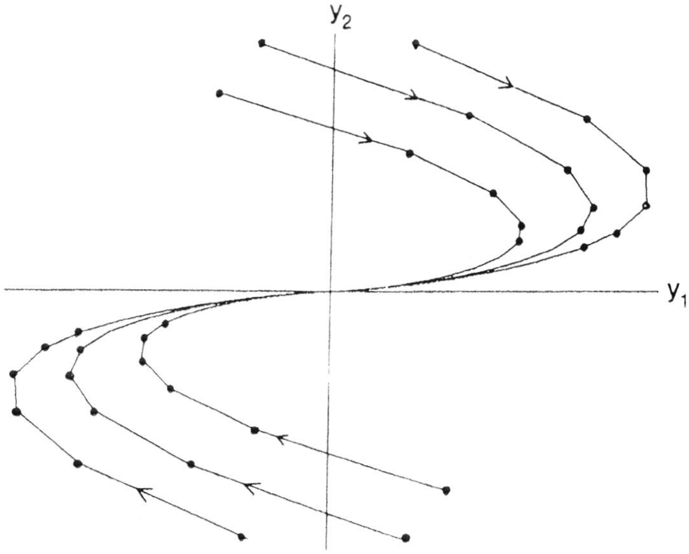


FIGURE 4.5a
(a) A degenerate sink: $\lambda_1 = \lambda_2 = \lambda$, $0 < \lambda < 1$.

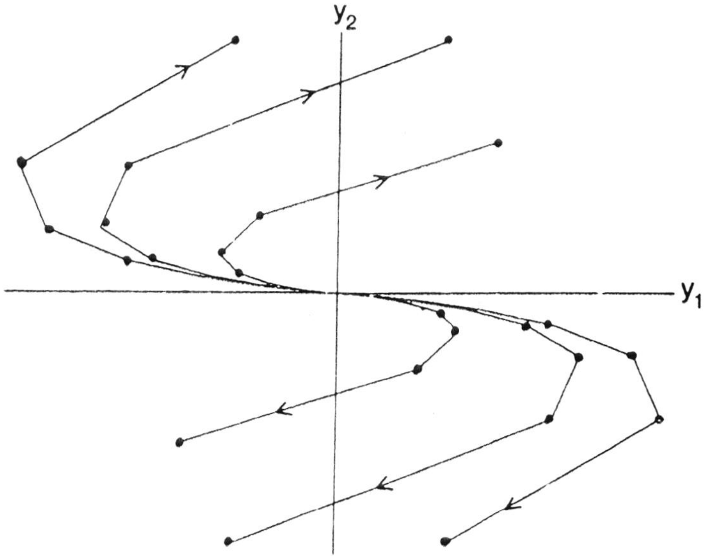


FIGURE 4.5b
(b) A degenerate source: $\lambda_1 = \lambda_2 = \lambda$, $\lambda > 1$.

degenerate sink. Figure 4.5b depicts the case when $|\lambda| > 1$ and in this case, the origin is called a degenerate source.

(III). Suppose that J is in the form

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

In this case, we have no straight line solutions due to the presence of $\cos n\beta$ and $\sin n\beta$ in the solutions $Y_1(n) = |\lambda_1|^n(k_1 \cos n\beta + k_2 \sin n\beta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y_2(n) = |\lambda_1|^n(-k_1 \sin n\beta + k_2 \cos n\beta)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The general solution is given by

$$Y(n) = |\lambda_1|^n \begin{pmatrix} k_1 \cos n\beta + k_2 \sin n\beta \\ -k_1 \sin n\beta + k_2 \cos n\beta \end{pmatrix}$$

with $Y(0) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$. Define an angle γ by setting $\cos\gamma = k_1/r_0$, and $\sin\gamma = k_2/r_0$, where $r_0 = \sqrt{k_1^2 + k_2^2}$. Then

$$\begin{aligned} y_1(n) &= |\lambda_1|^n r_0 \cos(n\omega - \gamma) \\ y_2(n) &= -|\lambda_1|^n r_0 \sin(n\omega - \gamma). \end{aligned}$$

Thus, the solution in polar coordinates is given by

$$\begin{aligned} r(n) &= \sqrt{y_1^2(n) + y_2^2(n)} \\ &= r_0|\lambda_1|^n \end{aligned} \tag{4.33}$$

$$\begin{aligned} \theta(n) &= \arctan\left(\frac{y_2(n)}{y_1(n)}\right) \\ &= -(nw - \gamma) \end{aligned} \tag{4.34}$$

It follows from Eqs. (4.33) and (4.34) that

1. If $|\lambda_1| < 1$, then we have a stable focus where each orbit spirals toward the origin [Fig. 4.6(a)]. On the other hand, if $|\lambda_1| > 1$, then we have an unstable focus, where each orbit spirals away from the origin [Fig. 4.6(b)].

2. If $|\lambda_1| = 1$, then we have a center, where the orbits follow a circular path [Fig. 4.6(c)]. This is due to the fact that $y_1^2(n) + y_2^2(n) = r_0^2$.

To this end, we have obtained the phase portraits of Equation (4.31), which may be called "canonical" phase portraits. To obtain the phase portraits of the original system (4.29), we apply (4.30), i.e., we apply P to the orbits of Equation (4.31). Since $P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (V_1, V_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = V_1$, and $P\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$

$(V_1, V_2)\begin{pmatrix}0\\1\end{pmatrix} = V_2$, applying P to the orbits $Y(n)$ amounts to rotating the coordinates; the x axis to V_1 and the y axis to V_2. In other words, the straight line solutions are now along the eigenvectors V_1 and V_2. Using this observation, one may opt to sketch the phase portrait of Equation (4.29) directly and without going through the canonical forms.

The set of points on the line emanating from the origin along V_1 is called the <u>stable</u> subspace W^s; the set of points on the line passing through the origin in the direction of V_2 is called the <u>unstable</u> subspace W^u. Hence,

$$W^s = \{X \in \mathbb{R}^2 : A^n X \to 0 \text{ as } n \to \infty\}, \tag{4.35}$$

$$W^u = \{X \in \mathbb{R}^2 : A^{-n} X \to 0 \text{ as } n \to \infty\} \tag{4.36}$$

The following example illustrates the above-described direct method to sketch the phase portrait.

Example 4.5

Sketch the phase portrait of the system $X(n+1) = AX(n)$, where

$$A = \begin{pmatrix}1 & 1\\0.25 & 1\end{pmatrix} \quad \square$$

SOLUTION The eigenvalues of A are $\lambda_1 = \frac{3}{2}$, and $\lambda_2 = \frac{1}{2}$; the corresponding eigenvectors are $V_1 = \begin{pmatrix}2\\1\end{pmatrix}$ and $V_2 = \begin{pmatrix}2\\-1\end{pmatrix}$, respectively. Hence, we have two straight line solutions, $X_1(n) = (1.5)^n\begin{pmatrix}2\\1\end{pmatrix}$ and $X_2(n) = (0.5)^n\begin{pmatrix}2\\-1\end{pmatrix}$. The general solution is given by $X(n) = k_1(1.5)^n\begin{pmatrix}2\\1\end{pmatrix} + k_2(0.5)^n\begin{pmatrix}2\\-1\end{pmatrix}$. Note that $x(n)$ is asymptotic to the line through $V_1 = \begin{pmatrix}2\\1\end{pmatrix}$ (see Fig. 4.7). ∎

4.6 Stability Notions

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and let $X^* = \begin{pmatrix}x_1^*\\x_2^*\end{pmatrix}$ be a fixed point of f; i.e., $f(X^*) = X^*$.

Our main objective in this section is to introduce the main stability notions pertaining to the fixed point x^*. Observe that these notions were previously

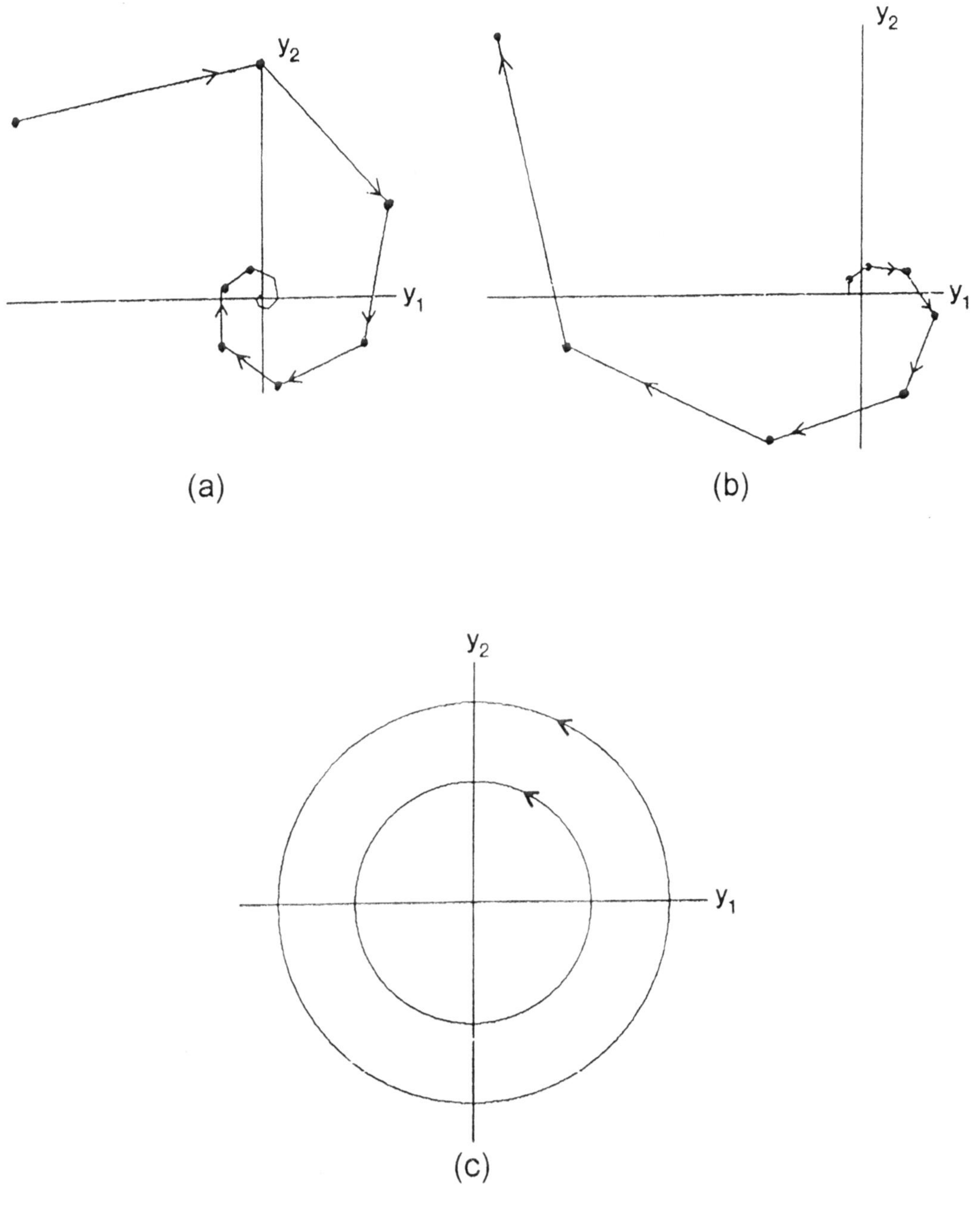


FIGURE 4.6
(a) Stable focus: $|\lambda_1| < 1$. (b) Source: $|\lambda_1| > 1$. (c) Center: $\lambda_{1,2} = \alpha \pm i\beta$, $|\lambda_{1,2}| = 1$.

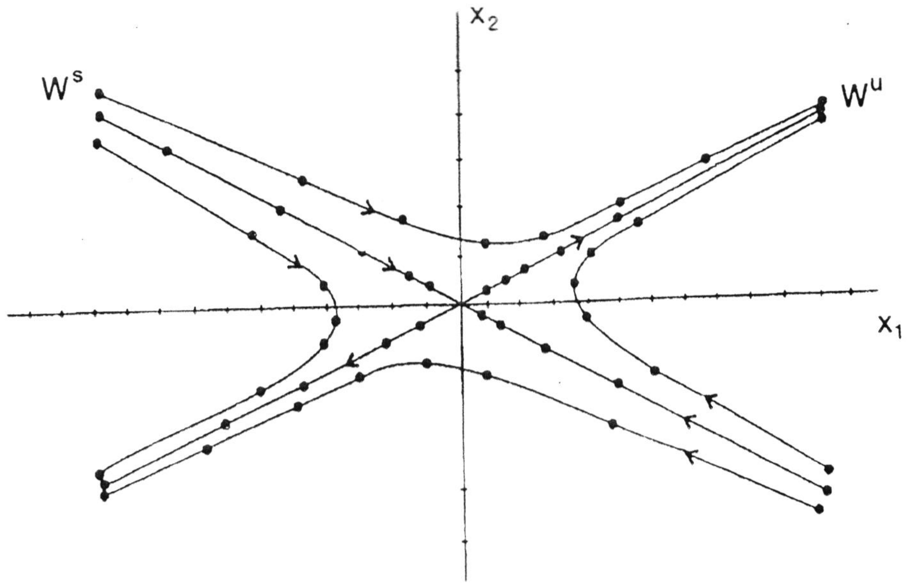


FIGURE 4.7
Saddle: $\lambda_1 > 1$, $0 < \lambda_2 < 1$. Stable and unstable subspaces: W^s, W^u.

introduced in Chapter 1. The only difference in $\mathbb{R}^2$ is that we replace the absolute value by a convenient "norm" on $\mathbb{R}^2$. Roughly speaking, a norm of a vector (point) in $\mathbb{R}^2$ is a measure of its magnitude. A formal definition follows.

DEFINITION 4.3 *A real valued function on a vector space V is said to be a norm on V, denoted by $|\ |$, if the following properties hold:*

1. $|X| \geq 0$ *and* $|X| = 0$ *if and only if* $X = 0$, *for* $X \in V$.
2. $|\alpha X| = |\alpha||X|$ *for* $X \in V$ *and any scalar* α.
3. $|X+Y| \leq |X| + |Y|$ *for* $X, Y \in V$ *(the triangle inequality).*

In the sequel, we choose the ℓ_1 norm on $\mathbb{R}^2$ defined for $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as

$$|X| = |x_1| + |x_2| \tag{4.37}$$

For each vector norm on $\mathbb{R}^2$ there corresponds a norm $||\ ||$ on all 2×2 matrices $A = (a_{ij})$ defined as follows

$$||A|| = \sup\{|AX| : |X| \leq 1\}. \tag{4.38}$$

It may be easily shown that for $X \in \mathbb{R}^2$,

$$|AX| \leq ||A||\ |X| \tag{4.39}$$

Let $\rho(A)$ be the **spectral radius** of A defined as $\rho(A) = \max\{|\lambda_1|, |\lambda_2| : \lambda_1, \lambda_2$ are the eigenvalues of $A\}$. Then it may be shown that for our selected vector norm

$$||A||_1 = \max\{(|a_{11}| + |a_{21}|), (|a_{12}| + |a_{22}|)\}. \tag{4.40}$$

(For a proof see [49].)

For example, if $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $|X| = 3$. And for the matrix $A = \begin{pmatrix} 1 & 3 \\ -2 & -4 \end{pmatrix}$, $||A||_1 = \max\{3, 7\} = 7$. The eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -1$. Note that $\rho(A) = \max\{|-2|, |-1|\} = 2$, and thus $\rho(A) \leq ||A||_1$.

It is left to the reader to prove, in general, that $\rho(A) \leq ||A||_1$ for any matrix A (Problem 14).

Without any further delay, we now give the required stability definitions.

DEFINITION 4.4 *A fixed point X^* of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be*

1. **stable** *if given $\varepsilon > 0$ there exists $\delta > 0$ such $|X - X^*| < \delta$ implies $|f^n(X) - X^*| < \varepsilon$ for all $n \in \mathbb{Z}^+$ (see Fig. 4.8a).*

2. **attracting** *(sink) if there exists $\nu > 0$ such that $|X - X^*| < \nu$ implies $\lim_{n\to\infty} f^n(X) = X^*$. It is* **globally attracting** *if $\nu = \infty$ (see Fig. 4.9).*

3. **asymptotically stable** *if it is both stable and attracting. It is* **globally asymptotically stable** *if it is both stable and globally attracting, (see Fig. 4.12(a))*

4. **unstable** *if it is not stable (see Fig. 4.8b).*

REMARK 4.2 In [91], Sedaghat showed that a globally attracting fixed point of a continuous one-dimensional map must be stable. Kenneth Palmer pointed out to me that this result may be found in the book of Block and Coppel [12]. Moreover, the proof in Block and Coppel requires only local attraction (see Appendix for a proof). The situation changes dramatically in two- or higher dimensional continuous maps, for there are continuous maps that possess a globally attracting unstable fixed point. We are going to present one of these maps and put several others as problems for you to investigate. ∎

Example 4.6

Consider the two-dimensional map in polar coordinates

$$g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{r} \\ \sqrt{2\pi\theta} \end{pmatrix}, \quad r > 0, 0 \leq \theta \leq 2\pi.$$

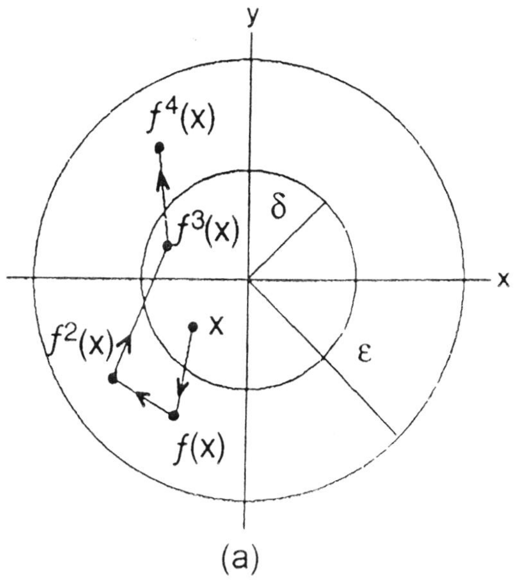


FIGURE 4.8a
(a) The fixed point $X^* = 0$ is stable.

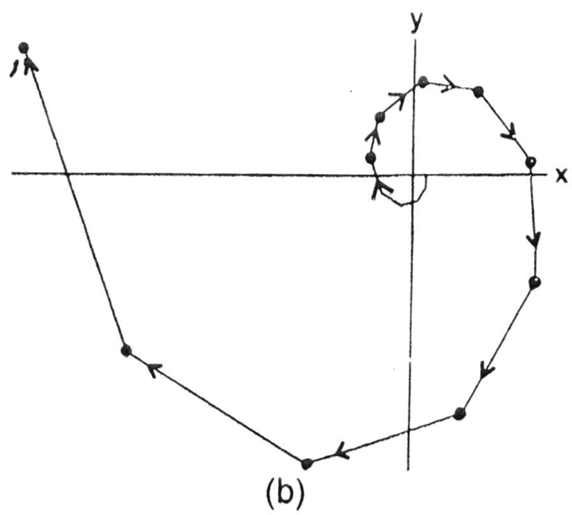


FIGURE 4.8b
(b) $X^* = 0$ is an unstable fixed point.

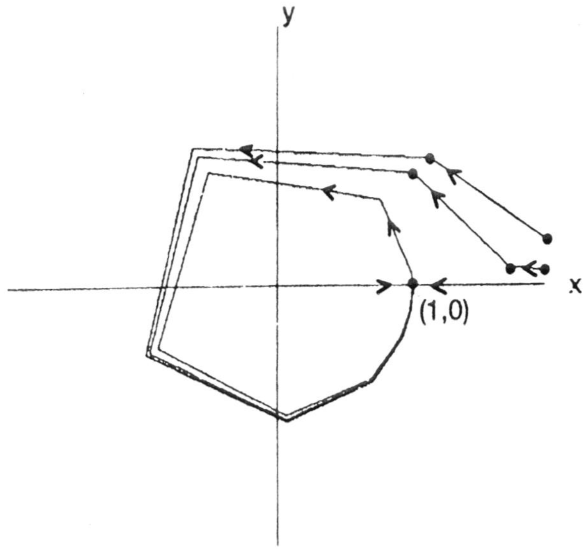


FIGURE 4.9
$x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an unstable globally attracting fixed point.

Then,

$$g^n \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r^{2^{-n}} \\ (2\pi)^{(1-2^{-n})} \theta^{2^{-n}} \end{pmatrix}.$$

Clearly $\lim_{n\to\infty} g^n \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 2\pi \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus, each orbit is attracted to the fixed point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. However, if $\theta = \delta\pi$, $0 < \delta < 1$, then the orbit of $\begin{pmatrix} r \\ \theta \end{pmatrix}$ will spiral clockwise around the fixed point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ before converging to it. Hence, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is globally attracting but not asymptotically stable (see Fig. 4.9). $\square$

4.7 Stability of Linear Systems

In this section, we focus our attention on linear maps where $f(X) = AX$, and A is a 2×2 matrix. Equivalently, we are interested in the difference equation

$$X(n+1) = AX(n). \tag{4.41}$$

For such linear maps, we can provide complete information about the stability of the fixed point $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The main result now follows.

THEOREM 4.3
The following statements hold for Equation (4.41):

(a) *If $\rho(A) < 1$, then the origin is asymptotically stable.*

(b) *If $\rho(A) > 1$, then the origin is unstable.*

(c) *If $\rho(A) = 1$, then the origin is unstable if the Jordan form is of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, and stable otherwise.*

PROOF Suppose that $\rho(A) < 1$. Then it follows from Eqs. (4.18), (4.19), (4.20), that $\lim_{n\to\infty} X(n) = 0$. Thus, the origin is (globally) attracting. To prove stability, we consider three cases.

(i) Suppose that the solution $X(n)$ is given by Equation (4.18). This is the case when the eigenvalues of the matrix A are real and there are two linearly independent eigenvectors.

$$X(n) = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} X(0).$$

Hence,

$$\begin{aligned} |X(n)| &\le ||P||\ ||P^{-1}|| \rho(A) |X(0)| \\ &\le M|X(0)| \end{aligned}$$

where $M = ||P||\ ||P^{-1}||\ \rho(A)$.

Now, given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then $|X(0)| < \delta$ implies that $|X(n)| < M\delta = \varepsilon$. This shows that the origin is stable.

(ii) Suppose that the solution $X(n)$ is given by Equation (4.19). This case occurs if the matrix A has a repeated eigenvalue λ and only one eigenvector.

$$X(n) = P \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} P^{-1} X(0)$$

$$|X(n)| \le ||P||\ ||P^{-1}||\ (n|\lambda|^{n-1} + |\lambda|^n)|X(0)|.$$

Since $n|\lambda|^n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{Z}^+$, such that the term $(n|\lambda|^{n-1} + |\lambda|^n)$ is bounded by a positive number L. Hence,

$$|X(n)| \le M|X(0)|$$

where $M = L||P||\ ||P^{-1}||$.

The proof of the stability of the origin may be completed by seting $\delta = \varepsilon/M$ as in part (a).

(iii) If A is not in the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then it is either diagonalizable to $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $|\lambda_1| < 1$ and $|\lambda_2| = 1$ or $|\lambda_2| < 1$ and $|\lambda_1| = 1$. In either case, J^n is bounded and hence the origin is stable. ∎

The proofs of parts (b) and (c) are left to you as Problem 11.

Exercises - (4.5 – 4.7)

صورة الطور

In Problems 1–9, sketch the phase portrait of the system $X(n+1) = AX(n)$, when A is the given matrix. Determine the stability of the origin.

1. $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

2. $\begin{pmatrix} 1/5 & 0 \\ 0 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

4. $\begin{pmatrix} -1/2 & 1 \\ 0 & -1/2 \end{pmatrix}$

5. $\begin{pmatrix} 0.5 & 0.25 \\ -0.25 & 0.5 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

7. $\begin{pmatrix} -3 & -4 \\ 7.5 & 8 \end{pmatrix}$

8. $\begin{pmatrix} -2 & 1 \\ -7 & 3 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{pmatrix}$

12. Show that if X^* is a fixed point of a linear map f on $\mathbb{R}^2$ and is asymptotically stable, then it must be globally asymptotically stable.

13. Complete the proof of Theorem 4.3.

14. Prove that for any 2×2 matrix $A, \rho(A) < ||A||$.

4.8 The Trace-Determinant Plane

4.8.1 Stability Analysis

Table (4.1) provides a partial summary of everything we have done so far. In this section we provide another important way of presenting these results, namely, trace-determinant plane, where we employ pictures, rather than a table. This turns out to be a better scheme when one is interested in studying bifurcation in two-dimensional systems.

The following two results provide the framework for using the trace-determinant plane. Recall that for matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $tr\ A = a_{11} + a_{22}$, and $det\ A = a_{11}a_{22} + a_{12}a_{21}$.

THEOREM 4.4
Let $A = (a_{ij})$ be a 2×2 matrix. Then $\rho(A) < 1$ if and only if

$$|tr\ A| - 1 < det\ A < 1. \tag{4.42}$$

PROOF

(i) Assume that $\rho(A) < 1$. If λ_1, λ_2 are the eigenvalues of A, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The characteristic equation of the matrix A is given by $det\ (A - \lambda I) = \lambda^2 - (a_{11} + a_{22})\lambda - (a_{11}a_{22} - a_{12}a_{21}) = 0$, or $\lambda^2 - (tr\ A)\lambda + det\ A = 0$. Hence the eigenvalues are

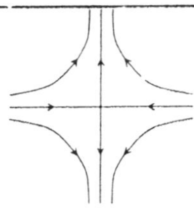
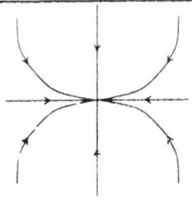
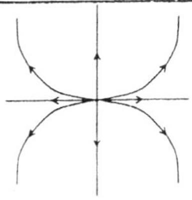
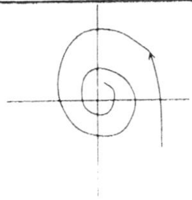
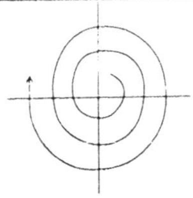
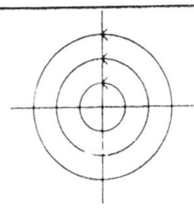
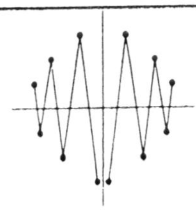
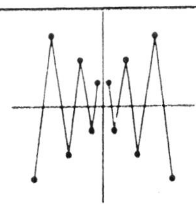
$$\lambda_1 = \frac{1}{2}\left[tr\ A + \sqrt{(tr\ A)^2 - 4\ det\ A}\right],$$

$$\lambda_2 = \frac{1}{2}\left[tr\ A - \sqrt{(tr\ A)^2 - 4\ det\ A}\right].$$

Case (a) λ_1 and λ_2 are real roots, i.e., $(tr\ A)^2 - 4\ det\ A \geq 0$. Now $-1 < \lambda_1$, $\lambda_2 < 1$ implies that

$$-2 < tr\ A \pm \sqrt{(tr\ A)^2 - 4\ det\ A} < 2$$

TABLE 4.1
Partial summary of the stability of linear systems.

Type	Eigenvalue	Phase Portrait
Saddle	$0 < \lambda_1 < 1 < \lambda_2$	
Sink	$0 < \lambda_2 < \lambda_1 < 1$	
Source	$\lambda_2 > \lambda_1 > 1$	
Spiral Sink	$\lambda = \alpha \pm i\beta,\ \lvert\lambda\rvert < 1,\ \beta \neq 0$	
Spiral Source	$\lambda = \alpha \pm i\beta,\ \lvert\lambda\rvert > 1,\ \beta \neq 0$	
Center	$\lambda = \alpha \pm i\beta,\ \lvert\lambda\rvert = 1,\ \beta \neq 0$	
"Oscillatory" Saddle	$-1 < \lambda_1 < 0,\ \lambda_2 < -1$	
"Oscillatory" Source	$\lambda_1 > 1,\ \lambda_2 > -1$	

or

$$-2 - tr\ A < \sqrt{(tr\ A)^2 - 4\ det\ A} < 2 - tr\ A \tag{4.43}$$

$$-2 - tr\ A < -\sqrt{(tr\ A)^2 - 4\ det\ A} < 2 - tr\ A. \tag{4.44}$$

Squaring the second inequality (4.43) yields

$$1 - tr\ A + det\ A > 0. \tag{4.45}$$

Similarly, if we square the first inequality in (4.44) we obtain

$$1 + tr\ A + det\ A > 0. \tag{4.46}$$

Now from the second inequality (4.43) and the first inequality in (4.44) we obtain $2 + tr\ A > 0$ and $2 - tr\ A > 0$ or $|tr\ A| < 2$. Since $(tr\ A)^2 - 4\ det\ A \geq 0$, it follows that

$$det\ A \leq (tr\ A)^2/4 < 1. \tag{4.47}$$

Combining (4.45), (4.46) and (4.47) yields (4.42).

Case (b) λ_1 and λ_2 are complex conjugates, i.e.,

$$(tr\ A)^2 - 4\ det\ A < 0. \tag{4.48}$$

In this case we have $\lambda_{1,2} = \frac{1}{2}\left[tr\ A \pm i\sqrt{4\ det\ A - (tr\ A)^2}\right]$ and

$$|\lambda_1|^2 = |\lambda_2|^2 = \frac{(tr\ A)^2}{4} + \frac{4\ det\ A}{4} - \frac{(tr\ A)^2}{4} = det\ A.$$

Hence $0 < det\ A < 1$. To show that inequalities (4.45) and (4.46) hold, note that since $det\ A > 0$ either (4.45) (if $tr\ A > 0$) or (4.46) (if $tr\ A < 0$) holds. Without loss of generality, assume that $tr\ A > 0$. Then (4.45) holds. From (4.48), $tr\ A < 2\sqrt{det\ A}$. If we let $det\ A = x$, then $x \in (0,1)$ and $f(x) = 1 + x - 2\sqrt{x} < 1 + det\ A - tr\ A$. Note that $f(0) = 1$ and $f'(0) = 1 - \frac{1}{\sqrt{x}}$ indicate that f is decreasing on $(0,1)$ with range $(0,1)$. This implies that $1 + det\ A - tr\ A > f(x) > 0$ and this completes the proof.

(ii) Conversely, assume that (4.42) holds. Then we have two cases to consider.

Case (a) $(tr\ A)^2 - 4\ det\ A \geq 0$. Then

$$\begin{aligned}
|\lambda_{1,2}| &= \frac{1}{2}\left|tr\ A \pm \sqrt{(tr\ A)^2 - 4\ det\ A}\right| \\
&< \frac{1}{2}\left|tr\ A \pm \sqrt{(det\ A + 1)^2 - 4\ det\ A}\right| \\
&< \frac{1}{2}\left(det\ A + 1 + \sqrt{(det\ A - 1)^2}\right) \\
&= \frac{1}{2}(det\ A + 1 - (det\ A - 1)) = 1.
\end{aligned}$$

Case (b) $(tr\ A)^2 - 4\ det\ A < 0$. Then

$$\begin{aligned}|\lambda_{1,2}| &= \frac{1}{2}\left|tr\ A \pm i\sqrt{4\ det\ A - (tr\ A)^2}\right| \\ &= \frac{1}{2}\sqrt{(tr\ A)^2 + 4\ det\ A - (tr\ A)^2} \\ &= \sqrt{det\ A} < 1.\end{aligned}$$

∎

As a by-product of the preceding result, we obtain the following criterion for asymptotic stability.

COROLLARY 4.1

The origin in Equation (4.41) is asymptotically stable if and only if condition (4.42) holds true.

Note that condition (4.42) may be spelled out in the following three inequalities:

$$1 + tr\ A + det\ A > 0, \quad \text{or}$$

$$\boxed{det\ A > -tr\ A - 1}(4.45)' \tag{4.49}$$

$$1 - tr\ A + det\ A > 0, \quad \text{or}$$

$$\boxed{det\ A > tr\ A - 1}(4.46)' \tag{4.50}$$

$$\boxed{det\ A < 1.}(4.47)' \tag{4.51}$$

Viewing $det\ A$ as a function of $tr\ A$, the above three inequalities give us the stability region as the interior of the triangle bounded by the lines $det\ A = -tr\ A - 1$, $det\ A = tr\ A - 1$, and $det\ A = 1$, as shown in Fig. 4.10.

Next we delve a little deeper into finding the exact values of the eigenvalues of the matrix A along the boundaries of the triangle enclosing the stability region. The following result provides us with the needed answers. Let $\lambda_1 = \frac{1}{2}\left(tr\ A + \sqrt{(tr\ A)^2 - 4\ det\ A}\right)$, $\lambda_2 = \frac{1}{2}\left(tr\ A - \sqrt{(tr\ A)^2 - 4\ det\ A}\right)$ be the eigenvalues of A.

THEOREM 4.5

The following statements hold for any 2×2 matrix A.

(i) If $|tr\ A| - 1 = det\ A$, then we have

(a) the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = det\ A$ if $tr\ A > 0$,

(b) the eigenvalues of A are $\lambda_2 = -1$ and $\lambda_1 = -det\ A$ if $tr\ A < 0$,.

(ii) If $|tr\ A| - 1 < det\ A$, and $det\ A = 1$, then the eigenvalues of A are $e^{\pm i\theta}$, where $\theta = \cos^{-1}(tr\ A/2)$.

PROOF

(i) Let $|tr\ A| - 1 = det\ A$. Then $(tr\ A)^2 - 4\ det\ A = (det\ A + 1)^2 \geq 0$. This implies that the eigenvalues are real numbers. Moreover, $\lambda_{1,2} = \frac{1}{2}\left(tr\ A \pm \sqrt{(tr\ A)^2 - 4\ det\ A}\right) = \frac{1}{2}(tr\ A \pm (det\ A - 1))$.

(a) If $tr\ A > 0$, then $tr\ A = 1 + det\ A$, and consequently,

$$\lambda_{1,2} = \begin{cases} 1 \\ det\ A \end{cases}.$$

(b) If $tr\ A < 0$, then $tr\ A = -1 - det\ A$, and consequently,

$$\lambda_{1,2} = \begin{cases} 1 \\ det\ A \end{cases}.$$

(ii) Let $|tr\ A| - 1 < det\ A$, and $det\ A = 1$. Then $(tr\ A)^2 - 4\ det A < (det\ A + 1)^2 - 4 = 0$. Hence, the eigenvalues are complex conjugates. Moreover,

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{2}\left((tr\ A)^2/4 \pm \sqrt{4\ det\ A - (tr\ A)^2}\right) \\ &= \frac{1}{2} tr\ A \pm \sqrt{1 - (tr\ A)^2}.\end{aligned}$$

Thus $|\lambda_{1,2}| = \sqrt{(tr\ A)^2/4 + 1 - (tr\ A)^2/4} = 1$. Furthermore, $\theta = \arctan(\lambda_{1,2})$ $= \tan^{-1}\left(\frac{\pm\sqrt{1 - (tr\ A)^2/4}}{(tr\ A)^2/4}\right) = cos^{-1}(tr\ A/2)$, which give $\lambda_{1,2} = e^{\pm i\theta} = \cos\theta \pm i \sin\theta$. ∎

4.8.2 Navigating the Trace-Determinant Plane

The trace-determinant plane is effective in the study of linear systems with parameters. It provides us a chart of those locations where we can expect dramatic changes in the phase portrait. There are three critical loci. Let T denotes the trace and D denote the determinant. Then there are three critical lines: $D = tr\ A - 1$, $D = -tr\ A - 1$, and $D = 1$; they enclose the stability region in the trace-determinant planes.

We now illustrate our analysis by the following example.

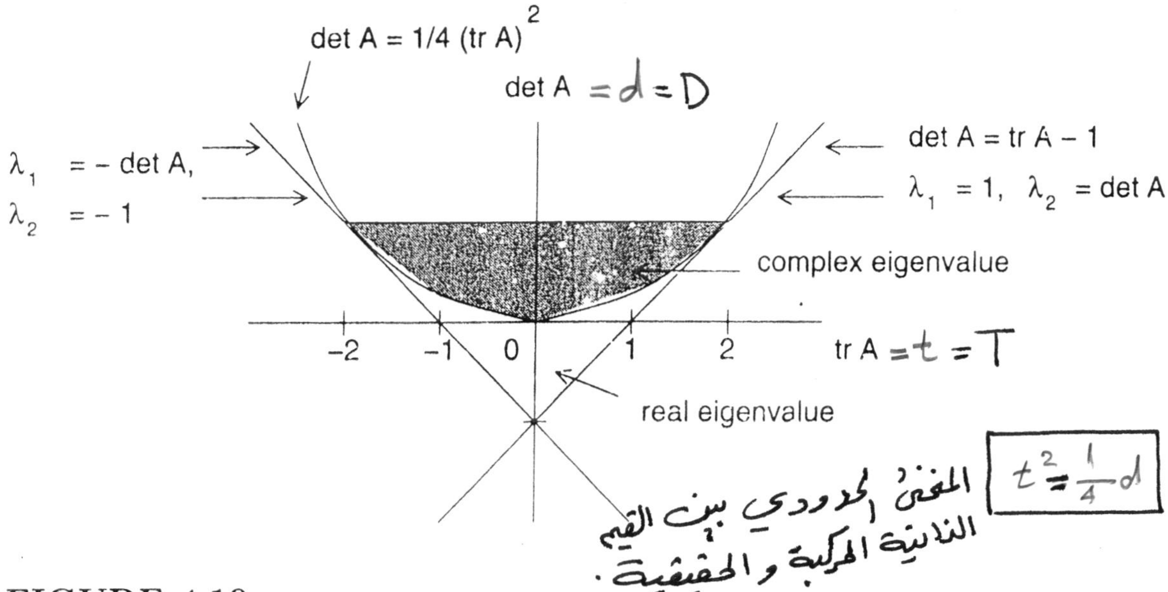


FIGURE 4.10a
(a) The stability region for Equation (4.41) is the shaded triangle.

$|T| - 1 < D < 1$

$|T| - 1 < D$ and $D < 1$

$|T| < D + 1$

$-D - 1 < T < D + 1$

$\rho(A) < 1 \Leftrightarrow$ قيود منطقة الاستقرارية المقاربة $\boxed{D > -T-1}$ and $\boxed{D > T-1}$ and $\boxed{D < 1}$

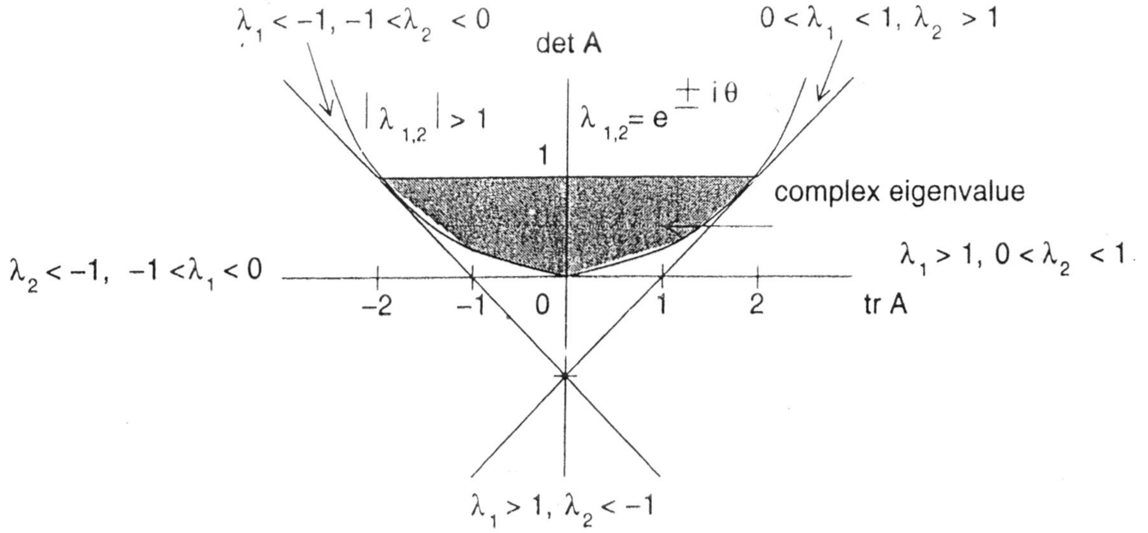


FIGURE 4.10b
(b) The determination of eigenvalues in different regions.

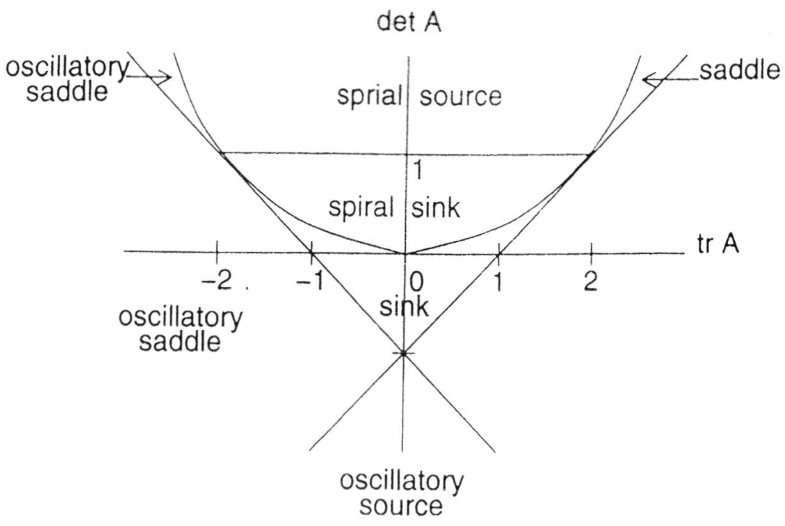


FIGURE 4.10c
(c) Description of the dynamics of Equation (4.41) in all the regions in the det-trace plane.

Example 4.7

Consider the one-parameter family of linear systems $X(n+1) = AX(n)$, where

$$A = \begin{pmatrix} -1 & a \\ -2 & 1 \end{pmatrix}$$

which depends on the parameter a. As a varies, the determinant of the matrix, *det* A, is always $2a-1$, while the trace of the matrix, *tr* A, is always 0. As we vary the parameter a from negative to positive values, the corresponding point (T, D) moves vertically along the line $T = 0$. Now if $D < -1$, which occurs if $2a - 1 < -1$ or $a < 0$, we have a degenerate case, $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus every point on the y-axis is a fixed point and every other point in the plane is periodic of period 2. For $0 < a \leq \frac{1}{2}$, we have a sink, and for $\frac{1}{2} < a < 1$ we have a spiral sink. At exactly $a = 1$ we have a center, and if $a > 1$ we have a spiral source (see Fig 4.10b).

The values of a where critical dynamical changes occur are called bifurcation values. In this example, the bifurcation values of a are 0, $\frac{1}{2}$, 1. $\square$

Exercises - 4.8

In problems 1-6, consider the one-parameter families of linear systems $X(n+1) = AX(n)$, where A depends on a parameter a. In a brief essay,

discuss different types of dynamical behavior exhibited by the systems as α increases along the real line, modeled after Example 4.7.

1. $A = \begin{pmatrix} 2 & 3 \\ 1+\alpha & 4 \end{pmatrix}$

2. $A = \begin{pmatrix} 3+\alpha & 2 \\ -2 & 3 \end{pmatrix}$

3. $A = \begin{pmatrix} 2+\alpha & 1 \\ 0 & 2 \end{pmatrix}$

4. $A = \begin{pmatrix} \alpha+1 & 1 \\ \alpha & \alpha+1 \end{pmatrix}$

5. $A = \begin{pmatrix} \alpha+1 & \alpha^2+\alpha \\ 1 & \alpha+1 \end{pmatrix}$

6. $A = \begin{pmatrix} \alpha+1 & \sqrt{1-\alpha^2} \\ 1 & 1 \end{pmatrix}$, $-1 \leq \alpha \leq 1$

In problems 7-9, we consider two-parameter families of the linear system $X(n+1) = AX(n)$, where A depends on two parameters α, β.

In the ab-plane, identify all regions where the system possesses a saddle, a sink, a spiral sink, and so on.

7.* $A = \begin{pmatrix} \alpha+1 & 1 \\ \beta & 2 \end{pmatrix}$

8.* $A = \begin{pmatrix} \alpha+1 & \beta \\ \beta & \alpha+1 \end{pmatrix}$

In the trace-determinant plane (Fig. 4.10d).

1. Show that we have a saddle in regions in ③ and ⑤.
2. Show that we have an oscillatory saddle in regions ⑦ and ⑧.
3. (a) Show that we have an oscillatory source in region ⑥.
   (b) Show that we have a spiral source in region ③.

X4.9 Liapunov Functions for Nonlinear Maps

In 1892, the Russian mathematician A. M. Liapunov (sometimes transliterated as Lyapunov) introduced a new method to investigate the stability of

Exercises - (4.1 and 4.2)

1. eigenvector $\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$ is associated with $\lambda_1 = 3$. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is associated with $\lambda_2 = \frac{1}{2}$, $A^n = \begin{pmatrix} -2 \cdot 3^n + \frac{3}{2^n} & 2 \cdot 3^n - \frac{1}{2^{n-1}} \\ -3^{n+1} + \frac{3}{2^n} & 3^{n+1} - \frac{1}{2^{n-1}} \end{pmatrix}$

3. $\lambda_1 = \lambda_2 = 2$

   $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

   $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   $A^n = \begin{pmatrix} \frac{2^n(n+3)}{3} & \frac{n \cdot 2^{n-1}}{3} \\ -\frac{n2^{n+1}}{3} & \frac{2n(3n-1)}{3} \end{pmatrix}$

5. $\lambda_1 = \frac{-1+\sqrt{3}i}{2}$, $\lambda_2 = \frac{-1-\sqrt{3}i}{2}$

   $v = \begin{pmatrix} -\frac{3}{2} + \frac{\sqrt{3}}{2}i \\ 1 \end{pmatrix}$

   $A_n = \begin{pmatrix} -\sqrt{3}\sin\frac{2n\pi}{3} + \cos\frac{2n\pi}{3} & -2\sqrt{3}\sin\frac{2n\pi}{3} \\ \frac{2\sqrt{3}}{3}\sin\frac{2n\pi}{3} & \cos\frac{2n\pi}{3} + \sqrt{3}\sin\frac{2n\pi}{3} \end{pmatrix}$

7. $X(n) = \begin{pmatrix} 2^{n-1}(n+2) \\ 2^n(1-n) \end{pmatrix}$

9. $f^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{3}\sin\frac{2n\pi}{3} \\ \cos\frac{2n\pi}{3} + \sqrt{3}\sin\frac{2n\pi}{3} \end{pmatrix}$

Exercises - (4.3 and 4.4)

1. $X(n) = \frac{1}{3}\begin{pmatrix} (-1)^n + 2^{n+1} \\ 2^{n+1} \end{pmatrix}$

3. $X(n) == \begin{pmatrix} k_1 2^n + k_2 3^n \\ k_1 2^n + 2k_2 3^n \end{pmatrix}$

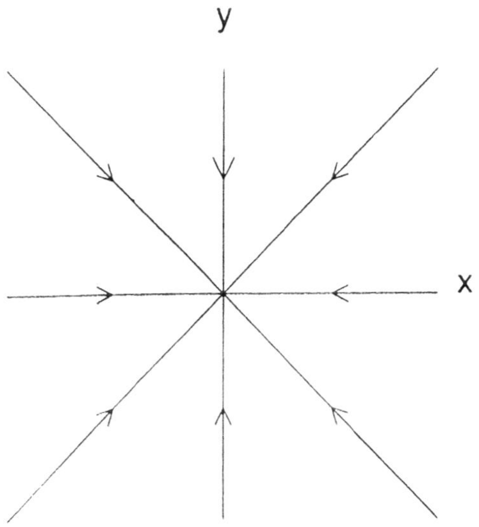
5. $X(n) == \begin{pmatrix} 3^n x_1(0) + 3^{n-1}x_2(0) \\ 3^n x_2(0) \end{pmatrix}$

7. $X(n) = k_1 \cos n\theta + k_2 \sin n\theta$ where $\theta = \tan^{-1}(4) \approx 76°$

15. $Y(n) == \begin{pmatrix} 2^n + n2^{n-1} - \frac{3}{4}n \\ 2^n - 1 \end{pmatrix}$

Exercises - (4.5 – 4.7)

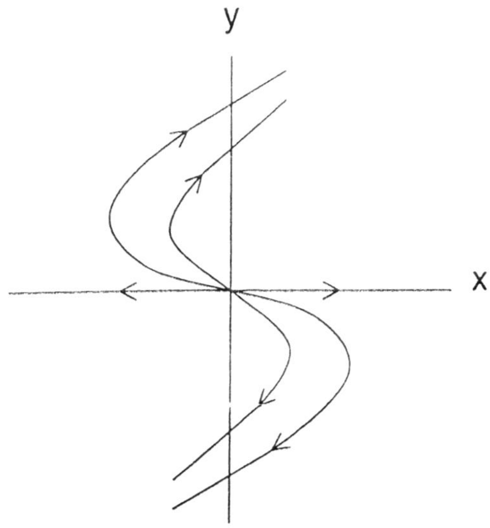
1. The origin is asymptotically stable since $\rho(A) = \frac{1}{2} < 1$ (Theorem 4.13).



$\lambda_1 = \lambda_2 = \frac{1}{2}$

Phase portrait: origin is asymptotically stable.

3. Since $\rho(A) = 2 > 1$, it follows by Theorem 4.13 that the origin is asymptotically stable.



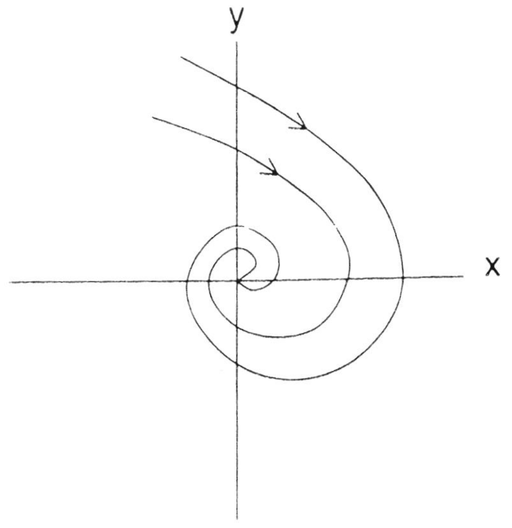
$\lambda_1 = \lambda_2 = 2$

Phase portrait: origin is unstable.

5. Eigenvalues of A are $\lambda_1 = \frac{1}{2} + \frac{1}{4}i$, $\lambda_2 = \frac{1}{2} - \frac{1}{4}i$.

$$|\lambda_1| = |\lambda_2| + \sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{\sqrt{5}}{4} < 1$$

Hence by Theorem 4.12, the origin is asymptotically stable.

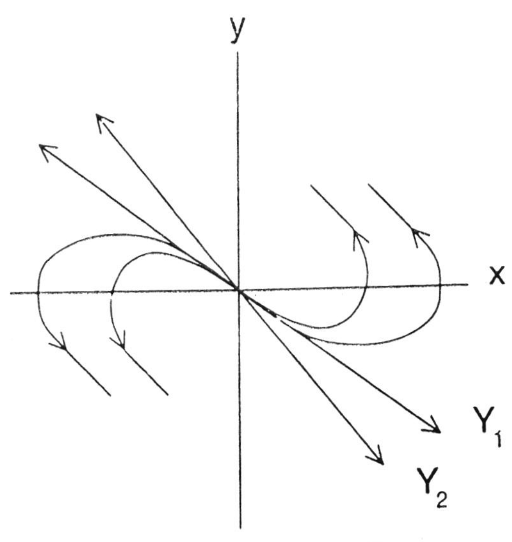


$\lambda_1 = \frac{1}{2} + \frac{1}{4}i, \quad \lambda_2 = \frac{1}{2} - \frac{1}{4}i$
A stable focus.

7. The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$. The corresponding eigenvectors are $V_1 = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $V_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, straight-line solutions are

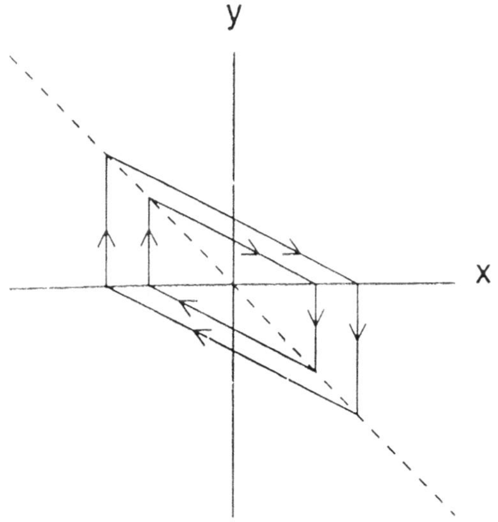
$$Y_1 = \begin{pmatrix} 4 \\ -5 \end{pmatrix} 2^n, \quad Y_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} 3^n.$$

The origin is unstable since $\rho(A) = 3 > 1$.



$\lambda_1 = 2, \quad \lambda_2 = 3$
Unstable node.

9. The eigenvalues of A are $\lambda_1 = i$, $\lambda_2 = -i$. The origin is stable but not asymptotically stable.



$\lambda_1 = i, \quad \lambda_2 = -i$
A stable center.

Exercises - 4.8

1. (i) $a > 4$: oscillatory source
   (ii) $a = 4$: unstable origin (bifurcation value)
   (iii) $a < 4$: a saddle
   (iv) $a = -\frac{4}{3}$: unstable origin
   (v) $a < -\frac{4}{3}$: spiral sources

3. Bifurcation value: $a = -3$
   $a < -3$: a saddle
   $a = -3$: unstable oscillatory

5. (i) $a >= -\frac{7}{6}$ where the eigenvalues are $\lambda_2 = -1$ and $\lambda_1 = -\det A = \frac{2}{3}$ and we have an "oscillatory" stable origin, for $a < -\frac{7}{6}$, we have an oscillatory saddle.
   (ii) $a = -1$, the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and every point is a fixed point, for $-\frac{7}{6} < a < -1$, we have a sink, and for $-1 < a < 0$, we have a spiral sink.

(iii) $a = 0$, where the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$, where we have eigher a stable or unstable origin, for $a > 0$, we have a saddle.

7. Region 5: $a < b$ and $3a + 6 > b$
   Region 6: $a < b$ and $3a + 6 < b$
   Region 7: $a > b$ and $3a + 6 < b$
   Region 1: $b > -\frac{1}{4}(a-1)^2$
   Region 8: $b < 3a + 6$ and $b < 2a + 1$
   Region 4: $b < a$ and $b > -\frac{1}{4}(a-1)^2$ and $b < 2a + 1$

Exercises - (4.9)

1. Let $V(x_1, x_2) = x_1^* + x_2^*$.

3. Let $V = x_1 x_2$.

5. Let $V(x, y) = x^2 + 4y^2$.

7. Let $V(x, y) = x^2 + y^2$.

11. Let $V(x, y) = xy$.

Exercises - (4.10 and 4.11)

3. $|\alpha| < 1$

5. $\alpha\beta < 1$

6. (a) $X_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable

   (b) $X_2^* = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ is unstable

7. $|b| < 1 - a < 2$